**Problem 1**) a) Using the method of integration by parts, $\int_{x_1}^{x_2} f'(x)g(x)\mathrm{d}x$ can be evaluated by setting $f'(x)=1$ and $g(x)=\ln(x)$ in the first integral and, similarly, $f'(x)=1$ and $g(x)=\ln^2(x)$ in the second. Other choices of $f'(x)$ and $g(x)$ are also possible.

$$\int_{x=1}^{x_0} \ln(x)\,\mathrm{d}x = x\ln(x)\big|_1^{x_0} - \int_{x=1}^{x_0} x(1/x)\mathrm{d}x = x_0\ln(x_0) - \ln(1) - (x_0-1)$$

$$= x_0\ln(x_0) - x_0 + 1.$$

$$\int_{x=1}^{x_0} \ln^2(x)\,\mathrm{d}x = x\ln^2(x)\big|_1^{x_0} - \int_{x=1}^{x_0} x(2/x)\ln(x)\,\mathrm{d}x = x_0\ln^2(x_0) - \ln^2(1) - 2\int_{x=1}^{x_0}\ln(x)\,\mathrm{d}x$$

$$= x_0\ln^2(x_0) - 2x_0\ln(x_0) + 2x_0 - 2.$$

Alternatively,

$$\int_{x=1}^{x_0} \ln^2(x)\,\mathrm{d}x = \int_{x=1}^{x_0} \ln(x)\ln(x)\,\mathrm{d}x = x[\ln(x)-1]\ln(x)\big|_1^{x_0} - \int_{x=1}^{x_0} x[\ln(x)-1](1/x)\mathrm{d}x$$

$$= x_0[\ln(x_0)-1]\ln(x_0) - [\ln(1)-1]\ln(1) - \int_{x=1}^{x_0}\ln(x)\,\mathrm{d}x + \int_{x=1}^{x_0}\mathrm{d}x$$

$$= x_0\ln^2(x_0) - x_0\ln(x_0) + 0 - [x_0\ln(x_0) - x_0 + 1] + (x_0 - 1).$$

$$= x_0\ln^2(x_0) - 2x_0\ln(x_0) + 2x_0 - 2.$$

b)
$$\mathrm{d}(\ln x)/\mathrm{d}x\big|_{x=1} = x^{-1}\big|_{x=1} = 1,$$

$$\mathrm{d}^2(\ln x)/\mathrm{d}x^2\big|_{x=1} = -x^{-2}\big|_{x=1} = -1,$$

$$\mathrm{d}^3(\ln x)/\mathrm{d}x^3\big|_{x=1} = 2x^{-3}\big|_{x=1} = 2,$$

$$\vdots$$

$$\mathrm{d}^n(\ln x)/\mathrm{d}x^n\big|_{x=1} = (n-1)!\,(-1)^{n-1}x^{-n}\big|_{x=1} = (-1)^{n-1}(n-1)!.$$

Consequently,

$$\ln(x) = \ln(1) + \textstyle\sum_{n=1}^{\infty}[\mathrm{d}^n(\ln x)/\mathrm{d}x^n|_{x=1}]x^n/n! = \sum_{n=1}^{\infty}(-1)^{n-1}(n-1)!\,(x-1)^n/n!$$

$$= \textstyle\sum_{n=1}^{\infty}(-1)^{n-1}(x-1)^n/n.$$

When $x\to 0$, we have $(x-1)^n \to (-1)^n$ and $\ln(x) \to \sum_{n=1}^{\infty}(-1)^{2n-1}/n = -\sum_{n=1}^{\infty}(1/n)$. Considering that $\sum_{n=1}^{\infty}(1/n)$ diverges to infinity, the value of $\ln(x)$ approaches $-\infty$ when $x\to 0$.

c)
$$\lim_{x_0\to 0}\int_{x=1}^{x_0}\ln(x)\,\mathrm{d}x = \lim_{x_0\to 0}[x_0\ln(x_0) - x_0 + 1] = 1 + \lim_{x_0\to 0}[x_0\ln(x_0)].$$

$$\int_{-\infty}^{0} e^y\mathrm{d}y = e^y\big|_{y=-\infty}^{0} 1 - 0 = 1.$$

Both of the above integrals represent the green-shaded area in the figure. Therefore,

$$1 + \lim_{x_0\to 0}[x_0\ln(x_0)] = 1 \quad\to\quad \lim_{x_0\to 0}[x_0\ln(x_0)] = 0.$$

d) The blue-shaded area in the figure is $\int_{x=1}^{x_0}\ln(x)\,\mathrm{d}x = x_0\ln(x_0) - x_0 + 1$. Similarly, the red-shaded area is $\int_{y=0}^{\ln(x_0)} e^y\mathrm{d}y = e^y\big|_{y=0}^{\ln(x_0)} = e^{\ln(x_0)} - e^0 = x_0 - 1$. Adding these two areas yields $x_0\ln(x_0)$, which is the area covered by the rectangle whose width and height are $x_0$ and $\ln(x_0)$.