Opti 403A/503A **Midterm Exam** (3/18/2026) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

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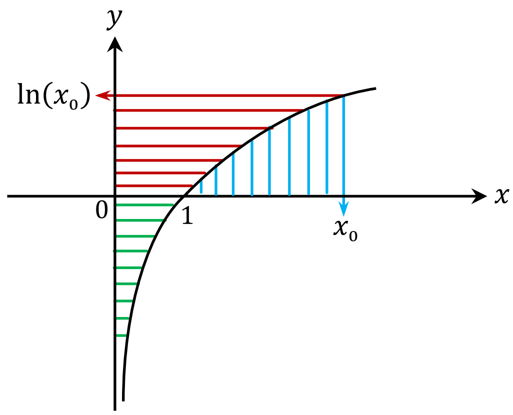
4 pts **Problem 1**) a) Use the method of integration by parts to evaluate the following integrals:

$$\int_{x=1}^{x_0} \ln(x)\,\mathrm{d}x, \qquad \int_{x=1}^{x_0} \ln^2(x)\,\mathrm{d}x.$$

3 pts b) Expand the function $\ln(x)$ in a Taylor series around the point $x = 1$. Show that, as $x \to 0$, the Taylor series approaches $-\sum_{n=1}^{\infty}(1/n)$. Conclude that $\ln(x)$ diverges to $-\infty$ as $x \to 0$.



3 pts c) The natural logarithm is the inverse of the exponential function. Stated differently, $y = \ln(x)$ implies that $x = e^y$. Thus, the green-shaded area bounded by (i) the negative y-axis, (ii) the $(0,1)$ segment of the x-axis, and (iii) the negative segment of $\ln(x)$ can be evaluated either as $\int_{x=1}^{0} \ln(x)\,\mathrm{d}x$ or as $\int_{y=-\infty}^{0} e^y \mathrm{d}y$. Use this equivalence to show that $\lim_{x_0 \to 0} x_0 \ln(x_0) = 0$.

3 pts d) Show that the blue-shaded area in the figure, which is the integral of $\ln(x)$ between $x = 1$ and $x = x_0 > 1$, plus the red-shaded area, which is the corresponding integral of e^y along the y-axis, equals the area of the rectangle having length x_0 and height $y_0 = \ln(x_0)$.

Problem 2) The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, \cdots$ follows the recurrence relation $f(n+2) = f(n) + f(n+1)$ starting at $n = 0$.

6 pts a) Find a closed form solution for $f(n)$, i.e., a compact formula that yields the Fibonacci numbers for any value of n.

6 pts b) Show that $F(N) = \sum_{n=0}^{N} f(n) = f(N+2) - 1$; that is, the sum of all Fibonacci numbers up to $n = N$ equals $f(N+2)$ minus 1.

Hint: The generic solution of the recurrence relation is $f(n) = \zeta^n$. Substitution into the recurrence relation yields two acceptable values, ζ_1 and ζ_2, for ζ. Write the general solution as $f(n) = A\zeta_1^n + B\zeta_2^n$, then find the unknown coefficients A and B by enforcing the initial conditions: $f(0) = 0$ and $f(1) = 1$. Part (b) can be solved by direct manipulation of the recurrence relation or, alternatively, by invoking the geometric series formula $\sum_{n=0}^{N} x^n = (x^{N+1} - 1)/(x - 1)$.

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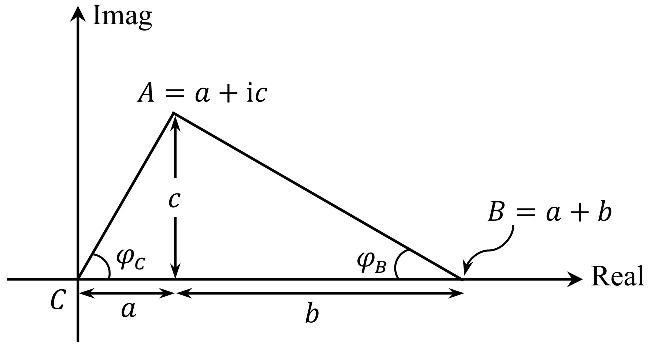
continued on the next page ⋯

10 pts **Problem 3**) Use complex-plane algebra to show that the altitudes of any triangle cross at a single point within the triangle's plane. The figure shows the complex-plane location of the vertex A as $a + \mathrm{i}c$, and that of the vertex B as $a + b$. The vertex C is at the origin of the complex plane.



Hint: Considering that $e^{\mathrm{i}\pi/2} = \mathrm{i}$, the complex number $A\mathrm{i} = -c + \mathrm{i}a$ is perpendicular to $\overline{AC}$. Similarly, $(A - B)\mathrm{i} = -c - \mathrm{i}b$ is perpendicular to $\overline{AB}$. Let s_1 and s_2 be arbitrary real numbers — positive, zero, or negative. The perpendicular dropped from B onto $\overline{AC}$ may then be expressed as $B + s_1 A\mathrm{i}$. Similarly, the perpendicular from C onto $\overline{AB}$ will be $s_2(A - B)\mathrm{i}$. Find s_1 and s_2 corresponding to the crossing point of these altitudes. Confirm that this crossing point lies on the perpendicular dropped from A onto $\overline{BC}$.