Problem 4) $f(x) = e^{sx}\sum_{n=0}^{\infty} a_n x^n,$

$$f'(x) = se^{sx}\sum_{n=0}^{\infty} a_n x^n + e^{sx}\sum_{n=1}^{\infty} a_n n x^{n-1},$$

$$f''(x) = s^2 e^{sx}\sum_{n=0}^{\infty} a_n x^n + 2se^{sx}\sum_{n=1}^{\infty} a_n n x^{n-1} + e^{sx}\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}.$$

Substitution into the differential equation yields

$$f''(x) + \gamma f'(x) + \omega_0^2 f(x) =$$

$$e^{sx}[(s^2 + \gamma s + \omega_0^2)\sum_{n=0}^{\infty} a_n x^n + (2s+\gamma)\sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}] = 0.$$

Changing the dummy of the second sum to $n' = n - 1$, and that of the third to $n' = n - 2$, one arrives at

$$(s^2 + \gamma s + \omega_0^2)\sum_{n=0}^{\infty} a_n x^n + (2s+\gamma)\sum_{n=0}^{\infty} a_{n+1}(n+1)x^n + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n = 0.$$

The above equation can be simplified by a judicious choice of s to eliminate either the first sum or the second sum. We examine each case separately.

Case i) $s^2 + \gamma s + \omega_0^2 = 0 \;\;\rightarrow\;\; s_\pm = -\tfrac{1}{2}\gamma \pm (\tfrac{1}{4}\gamma^2 - \omega_0^2)^{1/2} \;\;\rightarrow\;\; 2s_\pm + \gamma = \pm 2(\tfrac{1}{4}\gamma^2 - \omega_0^2)^{1/2}.$

To reduce clutter, define a new parameter $\zeta_\pm = \pm 2(\tfrac{1}{4}\gamma^2 - \omega_0^2)^{1/2}$, then write the remaining equation as follows:

$$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} + \zeta_\pm (n+1)a_{n+1}]x^n = 0.$$

The recursion relation is readily seen to be

$$a_{n+2} = -\left(\frac{\zeta_\pm}{n+2}\right)a_{n+1}, \qquad (n = 0, 1, 2, \cdots).$$

Note that, at this point, a_0 and a_1 are still arbitrary, and that the recursion relation yields the remaining coefficients $a_2, a_3, a_4, \cdots$. If $\zeta_\pm$ happens to be zero (i.e., $\omega_0 = \tfrac{1}{2}\gamma$, corresponding to the case of critical damping), then $a_2 = a_3 = \cdots = 0$ and $s_\pm = -\tfrac{1}{2}\gamma$, in which case we will have

$$f(x) = e^{sx}\sum_{n=0}^{\infty} a_n x^n = (a_0 + a_1 x)e^{-\frac{1}{2}\gamma x}.$$

In cases of overdamping and underdamping, where $\zeta_\pm \neq 0$, the recursion relation yields

$$\begin{aligned}
a_2 &= -(\zeta_\pm/2)a_1,\\
a_3 &= -(\zeta_\pm/3)a_2 = (\zeta_\pm^2/3!)a_1,\\
a_4 &= -(\zeta_\pm/4)a_3 = -(\zeta_\pm^3/4!)a_1,\\
a_5 &= -(\zeta_\pm/5)a_4 = (\zeta_\pm^4/5!)a_1.\\
&\;\;\vdots
\end{aligned}$$

Consequently,

$$\begin{aligned}
f(x) &= e^{sx}(a_0 + \textstyle\sum_{n=1}^{\infty} a_n x^n)\\
&= e^{s_\pm x}[a_0 + a_1\textstyle\sum_{n=1}^{\infty}(-\zeta_\pm)^{n-1}x^n/n!] = e^{s_\pm x}\left[a_0 - \frac{a_1}{\zeta_\pm}\left(e^{-\zeta_\pm x} - 1\right)\right]
\end{aligned}$$

$$= a_0 e^{s_\pm x} - \frac{a_1}{\zeta_\pm} e^{-\frac{1}{2}\gamma x}\left\{e^{\mp(\frac{1}{4}\gamma^2-\omega_0^2)^{\frac{1}{2}}x} - e^{\pm(\frac{1}{4}\gamma^2-\omega_0^2)^{\frac{1}{2}}x}\right\}$$

$$= e^{-\frac{1}{2}\gamma x}\left\{a_0 e^{\pm(\frac{1}{4}\gamma^2-\omega_0^2)^{\frac{1}{2}}x} + a_1 \frac{\sinh[(\frac{1}{4}\gamma^2-\omega_0^2)^{\frac{1}{2}}x]}{(\frac{1}{4}\gamma^2-\omega_0^2)^{\frac{1}{2}}}\right\}.$$

The above solution is a linear combination of $e^{s_\pm x}$ with arbitrary coefficients, which is the same as what one obtains via the conventional method of solving the differential equation.

Case ii) $2s + \gamma = 0 \quad \rightarrow \quad s = -\frac{1}{2}\gamma \quad \rightarrow \quad s^2 + \gamma s + \omega_0^2 = \omega_0^2 - \frac{1}{4}\gamma^2.$

The remaining equation now becomes

$$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} + (\omega_0^2 - \tfrac{1}{4}\gamma^2)a_n]x^n = 0.$$

The recursion relation is readily seen to be

$$a_{n+2} = \frac{\frac{1}{4}\gamma^2 - \omega_0^2}{(n+1)(n+2)} a_n, \qquad (n = 0, 1, 2, \cdots).$$

If $\omega_0 = \frac{1}{2}\gamma$ (i.e., the case of critical damping), a_0 and a_1 will be arbitrary, but $a_2 = a_3 = a_4 = \cdots = 0$. The general solution in this case will then be $f(x) = e^{sx}(a_0 + a_1 x) = (a_0 + a_1 x)e^{-\frac{1}{2}\gamma x}$, in agreement with the conventional solution obtained for the special case of critical damping.

If $\omega_0 \neq \frac{1}{2}\gamma$ (i.e., case of overdamping or underdamping), a_0 and a_1 will continue to be arbitrary, and the remaining coefficients will be

$$\begin{aligned} a_2 &= (\tfrac{1}{4}\gamma^2 - \omega_0^2)a_0/2!, & a_3 &= (\tfrac{1}{4}\gamma^2 - \omega_0^2)a_1/3!, \\ a_4 &= (\tfrac{1}{4}\gamma^2 - \omega_0^2)^2 a_0/4!, & a_5 &= (\tfrac{1}{4}\gamma^2 - \omega_0^2)^2 a_1/5!, \\ a_6 &= (\tfrac{1}{4}\gamma^2 - \omega_0^2)^3 a_0/6!, & a_7 &= (\tfrac{1}{4}\gamma^2 - \omega_0^2)^3 a_1/7!, \\ &\vdots & &\vdots \end{aligned}$$

Consequently,

$$f(x) = e^{sx}\sum_{n=0}^{\infty} a_n x^n = e^{-\frac{1}{2}\gamma x}\left[a_0 \sum_{n=0}^{\infty} \frac{(\frac{1}{4}\gamma^2-\omega_0^2)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(\frac{1}{4}\gamma^2-\omega_0^2)^n x^{2n+1}}{(2n+1)!}\right].$$

Recalling that $\sinh(x) = \sum_{n=0}^{\infty} x^{2n+1}/(2n+1)!$ and $\cosh(x) = \sum_{n=0}^{\infty} x^{2n}/(2n)!$, we will have

$$f(x) = e^{-\frac{1}{2}\gamma x}\left\{a_0 \cosh[(\tfrac{1}{4}\gamma^2 - \omega_0^2)^{\frac{1}{2}}x] + a_1 \frac{\sinh[(\frac{1}{4}\gamma^2-\omega_0^2)^{\frac{1}{2}}x]}{(\frac{1}{4}\gamma^2-\omega_0^2)^{\frac{1}{2}}}\right\}.$$

Once again, this solution contains a linear combination of $\exp[(-\frac{1}{2}\gamma \pm \sqrt{\frac{1}{4}\gamma^2 - \omega_0^2})x]$ with arbitrary coefficients, in agreement with the conventional solution of the differential equation.
