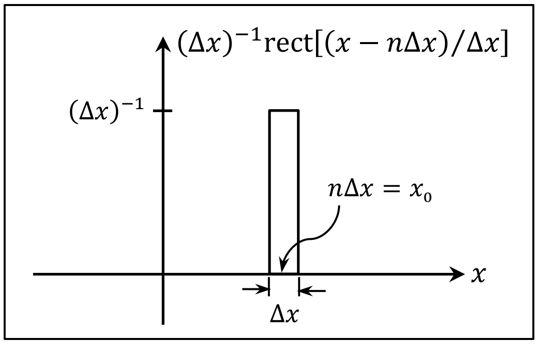
Problem 3) a) The function $(\Delta x)^{-1}\mathrm{rect}[(x-n\Delta x)/\Delta x]$ is a rectangular pulse of width Δx and height $(\Delta x)^{-1}$, centered at $x = n\Delta x$, as depicted in the figure below. In the limit when $\Delta x \to 0$ while $n = (x_0/\Delta x) \to \infty$, the rectangular pulse function approaches $\delta(x - x_0)$.



b) Multiplying the sampled values of $f(x)$ at $x = n\Delta x$ into rectangular pulses of width Δx centered at $x = n\Delta x$, we arrive at the following approximate form for $f(x)$:

$$f(x) \cong \sum_{n=-\infty}^{\infty} f(n\Delta x)\mathrm{rect}[(x-n\Delta x)/\Delta x]. \quad (1)$$

The multiplication of each $\mathrm{rect}(\cdot)$ function by $(\Delta x)^{-1}\Delta x$ does not change the above equation. However, in the limit when $\Delta x \to 0$ and $n\Delta x \to x'$, the properly scaled and shifted $\mathrm{rect}(\cdot)$ functions are replaced by shifted δ-functions, as follows:

$$f(x) = \lim_{\Delta x \to 0} \sum_{n=-\infty}^{\infty} f(n\Delta x)(\Delta x)^{-1}\mathrm{rect}[(x-n\Delta x)/\Delta x]\Delta x = \int_{-\infty}^{\infty} f(x')\delta(x-x')\mathrm{d}x'. \quad (2)$$

This, of course, is nothing more nor less than a convolution operation between $f(x)$ and $\delta(x)$. Note that Eq.(2) is consistent with the sifting property of the δ-function, considering that the value of $f(x')$ at the location $x' = x$ of the δ-function along the x'-axis is $f(x)$.

c) According to Eq.(2), the input function $f(x)$ is the sum of an infinite number of shifted δ-functions, namely, $\delta(x-x')$, each multiplied by $f(x')\mathrm{d}x'$. Since the system is shift-invariant, its response to $\delta(x-x')$ will be $h(x-x')$. The linearity of the system implies that its response to $f(x')\delta(x-x')\mathrm{d}x'$ will be $f(x')h(x-x')\mathrm{d}x'$. The sum of all the scaled δ-function inputs must, therefore, produce the sum of the corresponding outputs (again, due to the linearity of the system). Consequently,

$$g(x) = \int_{-\infty}^{\infty} f(x')h(x-x')\mathrm{d}x'. \quad (3)$$

The output $g(x)$ of an LSI system is thus seen to be the convolution between the input function, $f(x)$, and the impulse-response function, $h(x)$.