Problem 2) a) $f(x) = x^s \sum_{n=0}^{\infty} a_n x^n$,

$$f'(x) = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}.$$

Substitution into the differential equation yields

$$f'(x) - 2xf(x) = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1} - 2\sum_{n=0}^{\infty} a_n x^{n+s+1} \quad \boxed{n' = n+2}$$

$$= a_0 s x^{s-1} + a_1 (s+1) x^s + \sum_{n=2}^{\infty} a_n (n+s) x^{n+s-1} - 2\sum_{n=2}^{\infty} a_{n-2} x^{n+s-1} = 0.$$

The indicial equations and the recursion relation are thus found to be

$$a_0 s = 0,$$

$$a_1(s+1) = 0,$$

$$a_n = 2a_{n-2}/(n+s), \qquad (n = 2, 3, 4, \cdots).$$

Case i) $s = 0$. Here, a_0 is arbitrary, but $a_1 = 0$, and $a_n = (2/n)a_{n-2}$ for $n \geq 2$. Therefore, $a_1 = a_3 = a_5 = \cdots = 0$ and

$$a_2 = a_0, \qquad a_4 = ½a_2 = ½a_0, \qquad a_6 = ⅓a_4 = a_0/3!, \quad \cdots.$$

We thus have

$$f(x) = x^s \sum_{n=0}^{\infty} a_n x^n = a_0 \sum_{n=0}^{\infty} (x^{2n}/n!).$$

Case ii) $s = -1$. Here, a_1 is arbitrary, but $a_0 = 0$, and $a_n = 2a_{n-2}/(n-1)$ for $n \geq 2$. Therefore, $a_0 = a_2 = a_4 = \cdots = 0$, and

$$a_3 = a_1, \qquad a_5 = ½a_3 = ½a_1, \qquad a_7 = ⅓a_5 = a_1/3!, \quad \cdots.$$

We thus have

$$f(x) = x^s \sum_{n=0}^{\infty} a_n x^n = a_1 x^{-1} \sum_{n=0}^{\infty} (x^{2n+1}/n!) = a_1 \sum_{n=0}^{\infty} (x^{2n}/n!).$$

b) The solution is seen to be the same in both Cases (i) and (ii). This solution is indeed the Taylor series expansion of $f(x) = e^{x^2}$ around the point $x = 0$. This is because $e^x = \sum_{n=0}^{\infty} (x^n/n!)$, and when x is replaced by x^2, we obtain $e^{x^2} = \sum_{n=0}^{\infty} (x^{2n}/n!)$.

c) $\quad f'(x)/f(x) = 1/x^2 \quad \rightarrow \quad \ln[f(x)] = -x^{-1} + c \quad \rightarrow \quad f(x) = Ae^{-1/x}$.

Considering that the Taylor series expansion of e^x is $\sum_{n=0}^{\infty} (x^n/n!)$, it is seen immediately (upon replacing x with $-1/x$) that the corresponding expansion of $e^{-1/x}$ is $\sum_{n=0}^{\infty} (-1)^n/(n!\, x^n)$.

d) Applying the method of Frobenius to the differential equation, $x^2 f'(x) - f(x) = 0$, we find

$$f(x) = x^s \sum_{n=0}^{\infty} a_n x^n,$$

$$f'(x) = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}.$$

Substitution into the differential equation yields

$$x^2 f'(x) - f(x) = \sum_{n=0}^{\infty} a_n (n+s) x^{n+s+1} - \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$\boxed{n' = n+1}$$

$$= \sum_{n=1}^{\infty} a_{n-1} (n-1+s) x^{n+s} - \left(a_0 x^s + \sum_{n=1}^{\infty} a_n x^{n+s}\right) = 0.$$

The above equation is satisfied if $a_0 = 0$ and $a_n = (n + s - 1)a_{n-1}$ for $n = 1, 2, 3, \cdots$. This makes $a_0 = a_1 = a_2 = \cdots = 0$, irrespective of the value of s. The failure of the Frobenius method in the present example is rooted in the fact that the function $e^{-1/x}$ and all its derivatives vanish when x approaches zero from the positive side of the x-axis — they all go to infinity when $x \to 0$ from the negative side. As such, $f(x) = e^{-1/x}$ does not have a Taylor series expansion around $x = 0$. The presence of x^s in the Frobenius expression allows for a finite number of terms in the form of "x raised to a negative power" to appear in the presumed solution of the differential equation. However, a finite number of such terms is insufficient to generate a viable solution for the differential equation in the present case.