**Problem 1)**

a) $F(s) = \mathcal{F}\{\sin(\pi x)\} = (2\mathrm{i})^{-1}\int_{-\infty}^{\infty}(e^{\mathrm{i}\pi x} - e^{-\mathrm{i}\pi x})e^{-\mathrm{i}2\pi s x}\mathrm{d}x$

$$= (2\mathrm{i})^{-1}\left[\int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi(s-½)x}\mathrm{d}x - \int_{-\infty}^{\infty} e^{-\mathrm{i}2\pi(s+½)x}\mathrm{d}x\right]$$

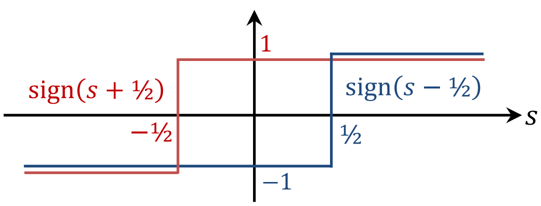
$$= (2\mathrm{i})^{-1}[\delta(s-½) - \delta(s+½)].$$

The Fourier transform of $g(x) = (\pi x)^{-1}$ is evaluated in the complex $z$-plane, with the large semi-circle closing in the upper-half-plane when $s < 0$, and in the lower-half-plane when $s > 0$. In both cases, Jordan's lemma guarantees the vanishing of the integral on the infinitely large semi-circle. The integrand has a single, first-order pole at $z = 0$, whose residue is $\pi^{-1}e^{-\mathrm{i}2\pi s z}\big|_{z=0} = \pi^{-1}$. Since the pole is located on the $x$-axis, it must be bypassed via an infinitesimally small semi-circle. In the case of $s < 0$, the small semi-circle will be in the upper-half-plane and traversed clockwise, thus contributing $\mathrm{i}\pi(\pi^{-1}) = \mathrm{i}$ to the integral. In the case of $s > 0$, the small semi-circle will be in the lower-half-plane and traversed counterclockwise, thus contributing $-\mathrm{i}\pi(\pi^{-1}) = -\mathrm{i}$ to the integral. All in all, we will have

$$G(s) = \mathcal{F}\{(\pi x)^{-1}\} = \int_{-\infty}^{\infty}(\pi x)^{-1}e^{-\mathrm{i}2\pi s x}\mathrm{d}x = \pi^{-1}\begin{cases}\mathrm{i}\pi, & s < 0\\ -\mathrm{i}\pi, & s > 0\end{cases} = -\mathrm{i}\,\mathrm{sign}(s).$$

Consequently,

$$\mathcal{F}\{\mathrm{sinc}(x)\} = F(s) * G(s) = -\mathrm{i}(2\mathrm{i})^{-1}[\mathrm{sign}(s-½) - \mathrm{sign}(s+½)] = \mathrm{rect}(s).$$




b) The scaling theorem of Fourier transformation yields $\mathcal{F}\{\alpha\,\mathrm{sinc}(\alpha x)\} = \mathrm{rect}(s/\alpha)$. In the limit of $\alpha \to \infty$, $\mathrm{rect}(s/\alpha)$ approaches 1 for all values of $s$. Consequently, $\mathcal{F}\{\delta(x)\} = 1(s)$. Parseval's theorem now yields $\int_{-\infty}^{\infty} h(x)\delta^*(x)\mathrm{d}x = \int_{-\infty}^{\infty} H(s)1^*(s)\mathrm{d}s = \int_{-\infty}^{\infty} H(s)\mathrm{d}s$. Since $\delta^*(x) = \delta(x)$, and the area under $H(s)$ equals $h(0)$, we will have $\int_{-\infty}^{\infty} h(x)\delta^*(x)\mathrm{d}x = h(0)$.

c) $\mathcal{F}\{\mathrm{cosc}(x)\} = ½[\delta(s-½) + \delta(s+½)] * [-\mathrm{i}\,\mathrm{sign}(s)]$

$$= -½\mathrm{i}[\mathrm{sgn}(s-½) + \mathrm{sign}(s+½)] = \begin{cases}\mathrm{i}, & s < -½;\\ 0, & -½ < s < ½;\\ -\mathrm{i}, & s > ½.\end{cases}$$

The scaling theorem of Fourier transformation now shows that $\mathcal{F}\{\alpha\,\mathrm{cosc}(\alpha x)\}$ is zero when $|s| < \alpha/2$, and $\pm\mathrm{i}$ when $|s| > \alpha/2$. Thus, in the limit of $\alpha \to \infty$, the Fourier transform of $\alpha\,\mathrm{cosc}(\alpha x)$ is zero everywhere. Invoking Parseval's theorem, it is now easy to verify that

$$\lim_{\alpha\to\infty}\int_{-\infty}^{\infty} h(x)[\alpha\,\mathrm{cosc}(\alpha x)]\mathrm{d}x = 0.$$

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