Opti 403A/503A **Final Exam** (5/12/2025) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

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5 pts **Problem 1**) a) The standard function $\operatorname{sinc}(x) = \sin(\pi x)/(\pi x)$ may be regarded as the product of $f(x) = \sin(\pi x) = (e^{\mathrm{i}\pi x} - e^{-\mathrm{i}\pi x})/2\mathrm{i}$ and $g(x) = (\pi x)^{-1}$. Compute the respective Fourier transforms $F(s)$ and $G(s)$, then show that $\mathcal{F}\{\operatorname{sinc}(x)\} = F(s) * G(s) = \operatorname{rect}(s)$.

5 pts b) Let $\delta(x) = \lim_{\alpha\to\infty}[\alpha \operatorname{sinc}(\alpha x)]$, in which case $\mathcal{F}\{\delta(x)\} = \lim_{\alpha\to\infty}[\operatorname{rect}(s/\alpha)] = 1$. Use Parseval's theorem to demonstrate the sifting property of $\delta(x)$, namely, $\int_{-\infty}^{\infty} h(x)\delta(x)\mathrm{d}x = h(0)$.

5 pts c) By analogy, define a new function $\operatorname{cosc}(x) = \cos(\pi x)/(\pi x)$, then use the same procedures as above to compute the Fourier transform of $\operatorname{cosc}(x)$. Argue that $\lim_{\alpha\to\infty}[\alpha \operatorname{cosc}(\alpha x)]$ is essentially a null (or zero) function of x, since its Fourier transform is zero at all frequencies.

Hint: To compute $F(s)$, you may invoke the known identity $\int_{-\infty}^{\infty} e^{\pm \mathrm{i}2\pi s x}\mathrm{d}x = \delta(s)$. Use complex-plane integration to compute $G(s)$. Parseval's theorem states that $\int_{-\infty}^{\infty} f(x)g^*(x)\mathrm{d}x = \int_{-\infty}^{\infty} F(s)G^*(s)\mathrm{d}s$.

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4 pts **Problem 2**) a) Use the method of Frobenius to solve the ordinary first-order differential equation

$$f'(x) - 2xf(x) = 0.$$

2 pts b) The equation may be equivalently written as $f'(x)/f(x) = 2x$. Integrating both sides of this equation, we find $\ln[f(x)] = x^2 + c$, which yields $f(x) = Ae^{x^2}$, with c being an integration constant, and $A = e^c$. Verify that this solution agrees with that obtained in part (a).

2 pts c) Solve the first-order ordinary differential equation $x^2 f'(x) - f(x) = 0$ using a direct method similar to that suggested in part (b). Proceed to expand this solution in a power series that resembles a Taylor series expansion, but consists entirely of the integer powers of $1/x$.

4 pts d) Show that the method of Frobenius does *not* yield a solution for the differential equation in part (c). Explain why the Frobenius method fails in this instance.

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3 pts **Problem 3**) a) Plot the function $(\Delta x)^{-1}\operatorname{rect}[(x - n\Delta x)/\Delta x]$, where Δx is a small length along the x-axis, and n is an arbitrary integer (positive, zero, or negative). What is the limiting form of this function when $\Delta x \to 0$ while $n \to \infty$, in such a way that $n\Delta x$ remains fixed at $x = x_0$?

5 pts b) A reasonably smooth but otherwise arbitrary function $f(x)$ can be approximated as a superposition of rectangular functions, as shown in the figure below. Write the approximate expression of $f(x)$ in terms of a sum (over n) of the $\operatorname{rect}(\cdot)$ functions described in part (a). Show that, in the limit when $\Delta x \to 0$, $f(x)$ can be expressed as the convolution of $f(x)$ with a delta-function. Confirm that the convolution integral in this case is consistent with the sifting property of the delta-function.

5 pts c) A linear and shift-invariant (LSI) system is defined as follows: Let the input $f_1(x)$ produce the output $g_1(x)$. Similarly, let the input $f_2(x)$ produce the output $g_2(x)$. Now, if the input is the superposition $af_1(x) + bf_2(x)$, where a and b are arbitrary constants, then the output will be $ag_1(x) + bg_2(x)$. Additionally, the system is shift-invariant if an arbitrarily-shifted input, say, $f(x - x_0)$, produces a correspondingly-shifted output $g(x - x_0)$.

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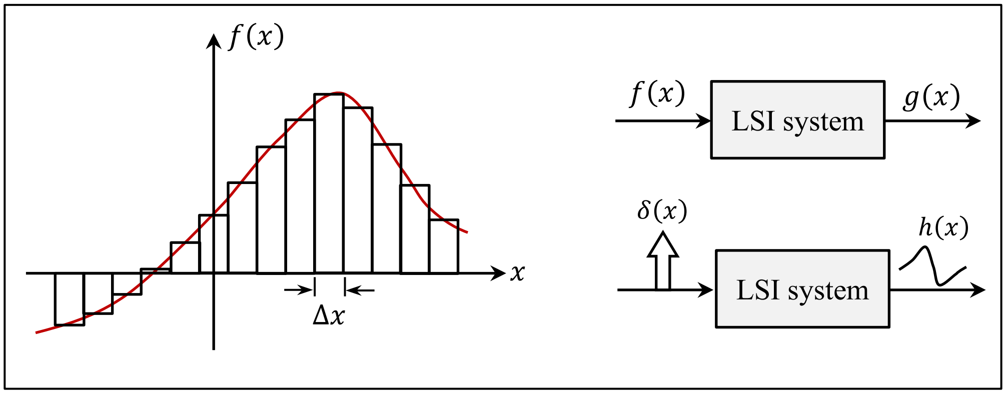
A characteristic feature of LSI systems is their so-called "impulse-response", which is the output $h(x)$ produced by the system in response to the input $\delta(x)$. Writing the input function $f(x)$ as a superposition of shifted δ-functions in accordance with part (b), express the output $g(x)$ of an LSI system in terms of its input $f(x)$ and its impulse-response function $h(x)$.



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10 pts **Problem 4**) A method similar to that of Frobenius for solving ordinary differential equations starts by assuming a solution in the form of $f(x) = e^{sx} \sum_{n=0}^{\infty} a_n x^n$, where the parameter s and the coefficients a_n are subsequently chosen to satisfy the differential equation. Use this method to solve the homogeneous, linear, 2nd-order, ordinary differential equation that governs harmonic oscillators (e.g., the mass-and-spring system), namely,

$$f''(x) + \gamma f'(x) + \omega_0^2 f(x) = 0.$$

Confirm that the solution thus obtained agrees with that obtained via the conventional method.

Hint: The resulting equation can be simplified in two different ways: (i) by setting $s^2 + \gamma s + \omega_0^2$ equal to zero, and (ii) by setting $2s + \gamma$ equal to zero. In each case, one must find the a_n coefficients accordingly.

Here, you are asked to consider only case (ii), where $2s + \gamma = 0$.
