Problem 5) Legendre equation: $(1-x^2)f''(x)-2xf'(x)+n(n+1)f(x)=0$.

Frobenius's solution: $f(x)=\sum_{k=0}^{\infty}A_k x^{k+s}$,

$$f'(x)=\sum_{k=0}^{\infty}(k+s)A_k x^{k+s-1},$$

$$f''(x)=\sum_{k=0}^{\infty}(k+s)(k+s-1)A_k x^{k+s-2}.$$

Substituting the above expressions into the Legendre equation, we find

$$\sum_{k=0}^{\infty}(k+s)(k+s-1)A_k x^{k+s-2}-\sum_{k=0}^{\infty}(k+s)(k+s-1)A_k x^{k+s}-\sum_{k=0}^{\infty}2(k+s)A_k x^{k+s}$$
$$+\sum_{k=0}^{\infty}n(n+1)A_k x^{k+s}=0.$$

The first sum in the above equation may be rewritten as follows:

$$\sum_{k=0}^{\infty}(k+s)(k+s-1)A_k x^{k+s-2}=s(s-1)A_0x^{s-2}+(1+s)sA_1x^{s-1}$$
$$+\sum_{k=2}^{\infty}(k+s)(k+s-1)A_k x^{k+s-2} \quad \boxed{k'=k-2}$$
$$=s(s-1)A_0x^{s-2}+s(s+1)A_1x^{s-1}+\sum_{k'=0}^{\infty}(k'+2+s)(k'+2+s-1)A_{k'+2}\,x^{k'+s}$$
$$=s(s-1)A_0x^{s-2}+s(s+1)A_1x^{s-1}+\sum_{k=0}^{\infty}(k+s+2)(k+s+1)A_{k+2}\,x^{k+s}. \quad \boxed{k'\to k}$$

Combining the preceding results, we arrive at

$$s(s-1)A_0=0,$$
$$s(s+1)A_1=0,$$

Indicial equations

$$(k+s+2)(k+s+1)A_{k+2}=[(k+s)(k+s-1)+2(k+s)-n(n+1)]A_k. \quad \boxed{k=0,1,2,\cdots}$$

Case i) $s=0$ satisfies both indicial equations, allowing A_0 and A_1 to be arbitrary coefficients. The recursion relation then yields the remaining coefficients, as follows:

$$A_{k+2}=\frac{k(k+1)-n(n+1)}{(k+1)(k+2)}A_k. \quad \boxed{k=0,1,2,\cdots}$$

Case ii) $s=1$ is a solution of the first indicial equation, which allows A_0 to be an arbitrary coefficient. However, A_1 must now be zero. The recursion relation in this case is

$$A_{k+2}=\frac{(k+1)(k+2)-n(n+1)}{(k+2)(k+3)}A_k. \quad \boxed{k=0,1,2,\cdots}$$

Consequently, $A_3=A_5=A_7=\cdots=0$. The nonzero coefficients obtained with the aid of the above recursion relation are $A_0, A_2, A_4, \cdots$. These are similar to the coefficients obtained in Case (i), corresponding to $s=0$, for $A_1, A_3, A_5, \cdots$, since k in the former recursion relation appears as $k+1$ in the latter. Considering that $f(x)=\sum_{k=0}^{\infty}A_{2k+1}x^{2k+1}$ in the case of $s=0$ with odd powers of x, whereas, in the case of $s=1$, $f(x)=x^s\sum_{k=0}^{\infty}A_{2k}x^{2k}=\sum_{k=0}^{\infty}A_{2k}x^{2k+1}$, we conclude that the present solution (obtained for $s=1$) is the same as the odd-powered solution obtained in Case (i) for $s=0$.

Case iii) $s=-1$ is a solution of the second indicial equation, which allows A_1 to be an arbitrary coefficient. However, A_0 must now be zero. The recursion relation in this case is

$$A_{k+2} = \frac{(k-1)k - n(n+1)}{k(k+1)} A_k. \quad \leftarrow \boxed{k = 0, 1, 2, \cdots}$$

The coefficients $A_1, A_3, A_5, \cdots$ obtained with the aid of the above recursion relation are similar to those obtained in Case (i), corresponding to $s = 0$, for $A_0, A_2, A_4, \cdots$, since k in the former recursion relation appears as $k - 1$ in the latter. Considering that $f(x) = \sum_{k=0}^{\infty} A_{2k} x^{2k}$ in the case of $s = 0$ with even powers of x, whereas, in the present case of $s = -1$, $f(x) = x^s \sum_{k=0}^{\infty} A_{2k+1} x^{2k+1} = \sum_{k=0}^{\infty} A_{2k+1} x^{2k}$, we conclude that the present solution (obtained in the case of $s = -1$) is the same as the even-powered solution obtained in Case (i) for $s = 0$.

In the case of $s = -1$ and $A_0 = 0$, the presence of k in the recursion relation's denominator indicates that $A_2 \neq 0$ should be allowed. Thus, $f(x) = x^s \sum_{k=1}^{\infty} A_{2k} x^{2k} = \sum_{k=1}^{\infty} A_{2k} x^{2k-1}$ is a legitimate solution. However, this is the same solution as that obtained in Case (i) for $s = 0$ with odd powers of x. Once again, a solution with $s = -1$ is subsumed by the solutions for $s = 0$.

Returning to the general solution obtained in Case (i) for $s = 0$, we now try to rewrite the numerator of the recursion relation as a *product* of two terms. The second-order polynomial $k(k+1) - n(n+1)$ has two roots; that is,

$$k^2 + k - n(n+1) = 0 \quad \rightarrow \quad k = -½ \pm \sqrt{¼ + n(n+1)} = -½ \pm \sqrt{(n+½)^2} = n, -(n+1).$$

Therefore, $k(k+1) - n(n+1) = (k-n)(k+n+1)$. The recursion relation thus becomes

$$A_{k+2} = \frac{(k-n)(k+n+1)}{(k+1)(k+2)} A_k. \quad \leftarrow \boxed{k = 0, 1, 2, \cdots}$$

For even coefficients, we have

$$A_2 = -\frac{n(n+1)}{1 \cdot 2} A_0,$$

$$A_4 = \frac{(2-n)(3+n)}{3 \cdot 4} \times \frac{-n(n+1)}{1 \cdot 2} A_0 = \frac{n(n-2) \times (n+1)(n+3)}{4!} A_0,$$

$$A_6 = -\frac{n(n-2)(n-4) \times (n+1)(n+3)(n+5)}{6!} A_0,$$

$$\vdots$$

$$A_{2m} = (-1)^m \frac{n(n-2)(n-4)\cdots(n-2m+2) \times (n+1)(n+3)(n+5)\cdots(n+2m-1)}{(2m)!} A_0.$$

Note that, if n is an even integer, the above series terminates at $k = 2m = n$, whereas for odd-integer values of n, the series continues indefinitely. Similarly, for odd coefficients, we find

$$A_3 = -\frac{(n-1)(n+2)}{2 \cdot 3} A_1,$$

$$A_5 = \frac{(3-n)(n+4)}{4 \cdot 5} \times \frac{-(n-1)(n+2)}{2 \cdot 3} A_1 = \frac{(n-1)(n-3) \times (n+2)(n+4)}{5!} A_1,$$

$$\vdots$$

$$A_{2m+1} = (-1)^m \frac{(n-1)(n-3)\cdots(n-2m+1) \times (n+2)(n+4)\cdots(n+2m)}{(2m+1)!} A_1.$$

Note that, if n is an odd integer, the above series terminates at $k = 2m + 1 = n$, whereas for even-integer values of n, the series continues indefinitely. The two *independent* solutions of the Legendre equation are thus given by

$$f_1(x) = A_0 + \sum_{m=1}^{\infty} A_{2m} x^{2m}, \qquad f_2(x) = A_1 x + \sum_{m=1}^{\infty} A_{2m+1} x^{2m+1}.$$
