Problem 4) a) $\int_0^{\frac{1}{2}\pi}\ln(\cos x)\,\mathrm{d}x \underset{\uparrow}{=} -\int_{\frac{1}{2}\pi}^{0}\ln[\cos(\tfrac{1}{2}\pi - y)]\,\mathrm{d}y = \int_0^{\frac{1}{2}\pi}\ln(\sin y)\,\mathrm{d}y.$

change of variable: $x = \tfrac{1}{2}\pi - y$

b)

$$\begin{aligned}
\int_0^{\frac{1}{2}\pi}\ln(\sin x)\,\mathrm{d}x &= \int_0^{\frac{1}{2}\pi}\ln[2\sin(\tfrac{1}{2}x)\cos(\tfrac{1}{2}x)]\,\mathrm{d}x \qquad \boxed{\text{change of variable: } x = \pi - y}\\
&= \int_0^{\frac{1}{2}\pi}\ln(2)\,\mathrm{d}x + \int_0^{\frac{1}{2}\pi}\ln[\sin(\tfrac{1}{2}x)]\,\mathrm{d}x + \int_0^{\frac{1}{2}\pi}\ln[\cos(\tfrac{1}{2}x)]\,\mathrm{d}x\\
&= \tfrac{1}{2}\pi\ln 2 + \int_0^{\frac{1}{2}\pi}\ln[\sin(\tfrac{1}{2}x)]\,\mathrm{d}x - \int_{\pi}^{\pi-\frac{1}{2}\pi}\ln[\cos\tfrac{1}{2}(\pi - y)]\,\mathrm{d}y\\
&= \tfrac{1}{2}\pi\ln 2 + \int_0^{\frac{1}{2}\pi}\ln[\sin(\tfrac{1}{2}x)]\,\mathrm{d}x + \int_{\frac{1}{2}\pi}^{\pi}\ln[\sin(\tfrac{1}{2}y)]\,\mathrm{d}y\\
&= \tfrac{1}{2}\pi\ln 2 + \int_0^{\pi}\ln[\sin(\tfrac{1}{2}y)]\,\mathrm{d}y \;\leftarrow \boxed{\text{change of variable: } x = \tfrac{1}{2}y}\\
&= \tfrac{1}{2}\pi\ln 2 + 2\int_0^{\frac{1}{2}\pi}\ln(\sin x)\,\mathrm{d}x.
\end{aligned}$$

Consequently,

$$\int_0^{\frac{1}{2}\pi}\ln(\sin x)\,\mathrm{d}x = -\tfrac{1}{2}\pi\ln 2.$$

c)

$$\begin{aligned}
\int_0^{\pi} x\ln(\sin x)\,\mathrm{d}x &= \int_0^{\frac{1}{2}\pi} x\ln(\sin x)\,\mathrm{d}x + \int_{\frac{1}{2}\pi}^{\pi} x\ln(\sin x)\,\mathrm{d}x \;\leftarrow \boxed{\text{change of variable: } x = \pi - y}\\
&= \int_0^{\frac{1}{2}\pi} x\ln(\sin x)\,\mathrm{d}x - \int_{\pi-\frac{1}{2}\pi}^{0}(\pi - y)\ln[\sin(\pi - y)]\,\mathrm{d}y\\
&= \int_0^{\frac{1}{2}\pi} \cancel{x\ln(\sin x)\,\mathrm{d}x} + \int_0^{\frac{1}{2}\pi}(\pi - \cancel{y)\ln(\sin y)\,\mathrm{d}y}\\
&= \pi\int_0^{\frac{1}{2}\pi}\ln(\sin y)\,\mathrm{d}y = -\tfrac{1}{2}\pi^2\ln 2. \;\leftarrow \boxed{\text{substitution from part (b)}}
\end{aligned}$$
