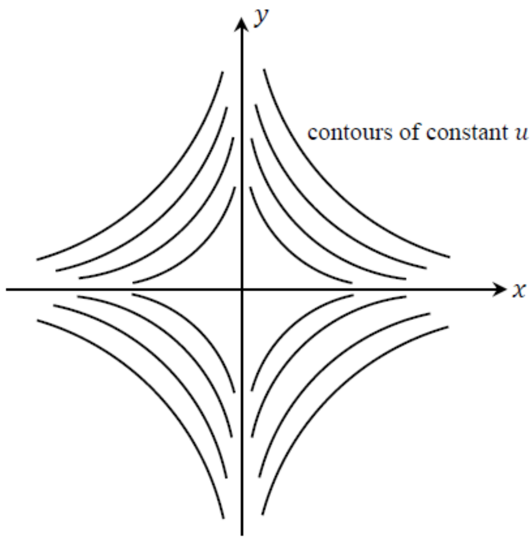
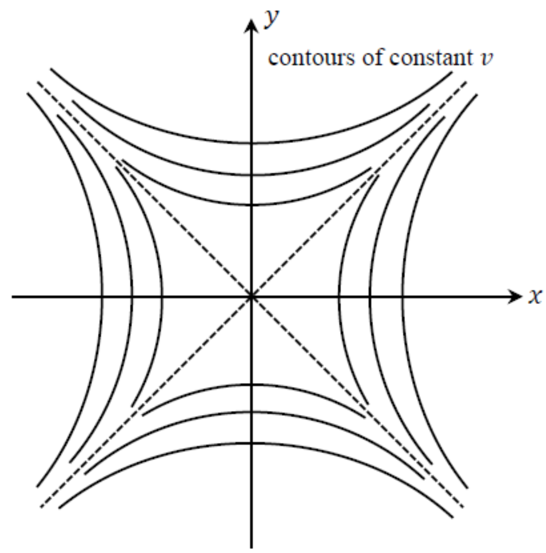
Problem 13) In the figure below, plots of constant u are shown on the left. For $u > 0$, the curves are in the first and third quadrants, for $u < 0$, the curves are in the second and fourth quadrants, and for $u = 0$, the curves coincide with the x and y axes. The plots of constant v are shown on the right. For $v > 0$, the branches of constant v are those on the right- and left-hand sides of the figure, for $v < 0$, the branches of constant v are those at the top and bottom of the figure, and for $v = 0$, the curves of constant v coincide with the straight lines $y = \pm x$.





Treating x and y as functions of u and v, that is, $x(u, v)$ and $y(u, v)$, we proceed to differentiate the curvilinear-coordinate equations with respect to u and v, as follows:

$$xy = u \quad \rightarrow \quad \begin{cases} \partial(xy)/\partial u = 1 \quad \rightarrow \quad (\partial x/\partial u)y + x(\partial y/\partial u) = 1, & (1) \\ \partial(xy)/\partial v = 0 \quad \rightarrow \quad (\partial x/\partial v)y + x(\partial y/\partial v) = 0. & (2) \end{cases}$$

$$x^2 - y^2 = v \quad \rightarrow \quad \begin{cases} \partial(x^2 - y^2)/\partial u = 0 \quad \rightarrow \quad 2x(\partial x/\partial u) - 2y(\partial y/\partial u) = 0, & (3) \\ \partial(x^2 - y^2)/\partial v = 1 \quad \rightarrow \quad 2x(\partial x/\partial v) - 2y(\partial y/\partial v) = 1. & (4) \end{cases}$$

Solving Eqs.(1) and (3) for $\partial x/\partial u$ and $\partial y/\partial u$, we find

$$\begin{pmatrix} y & x \\ x & -y \end{pmatrix} \begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} \partial x/\partial u \\ \partial y/\partial u \end{pmatrix} = \frac{1}{-y^2-x^2} \begin{pmatrix} -y & -x \\ -x & y \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\rightarrow \quad \partial x/\partial u = y/(x^2 + y^2), \qquad \partial y/\partial u = x/(x^2 + y^2). \qquad (5)$$

Similarly, solving Eqs.(2) and (4) for $\partial x/\partial v$ and $\partial y/\partial v$, we find

$$\begin{pmatrix} y & x \\ x & -y \end{pmatrix} \begin{pmatrix} \partial x/\partial v \\ \partial y/\partial v \end{pmatrix} = \begin{pmatrix} 0 \\ ½ \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} \partial x/\partial v \\ \partial y/\partial v \end{pmatrix} = \frac{1}{-y^2-x^2} \begin{pmatrix} -y & -x \\ -x & y \end{pmatrix} \begin{pmatrix} 0 \\ ½ \end{pmatrix}$$

$$\rightarrow \quad \partial x/\partial v = x/[2(x^2 + y^2)], \qquad \partial y/\partial v = -y/[2(x^2 + y^2)]. \qquad (6)$$

In terms of the above partial derivatives, the vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ are given by

$$\boldsymbol{U} = (\partial x/\partial u)\hat{\boldsymbol{x}} + (\partial y/\partial u)\hat{\boldsymbol{y}} = (y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}})/(x^2 + y^2), \tag{7}$$

$$\boldsymbol{V} = (\partial x/\partial v)\hat{\boldsymbol{x}} + (\partial y/\partial v)\hat{\boldsymbol{y}} = ½(x\hat{\boldsymbol{x}} - y\hat{\boldsymbol{y}})/(x^2 + y^2). \tag{8}$$

It is now easy to see that $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthogonal, because their dot-product vanishes; that is, $\boldsymbol{U} \cdot \boldsymbol{V} = ½(xy - yx)/(x^2 + y^2)^2 = 0$ for all values of x and y. The Jacobian of the coordinate transformation is found to be

$$\frac{\partial(x,y)}{\partial(u,v)} = \left\|\begin{matrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{matrix}\right\| = \left\|\begin{matrix} y/(x^2+y^2) & x/(x^2+y^2) \\ x/2(x^2+y^2) & -y/2(x^2+y^2) \end{matrix}\right\| = \frac{1}{2(x^2+y^2)}. \tag{9}$$

Note that the coordinate transformation equations yield

$$(x^2 - y^2)^2 + 4x^2y^2 = v^2 + 4u^2 \quad \rightarrow \quad x^2 + y^2 = \sqrt{4u^2 + v^2}. \tag{10}$$

Consequently, as a function of the u and v coordinates, the Jacobian in Eq.(9) is $1/2\sqrt{4u^2 + v^2}$.

Digression 1. The Jacobian can also be computed for the inverse transformation, i.e., from the uv to the xy coordinates. The result will be the inverse of the Jacobian given by Eq.(9), namely,

$$\frac{\partial(u,v)}{\partial(x,y)} = \left\|\begin{matrix} \partial u/\partial x & \partial v/\partial x \\ \partial u/\partial y & \partial v/\partial y \end{matrix}\right\| = \left\|\begin{matrix} y & 2x \\ x & -2y \end{matrix}\right\| = 2(x^2 + y^2). \tag{11}$$

Digression 2. For a direct calculation of the Jacobian, one must express x and y as functions of u and v. Substitution of $y = u/x$ in the equation $x^2 - y^2 = v$ yields

$$x^2 - (u/x)^2 = v \quad \rightarrow \quad x^4 - vx^2 - u^2 = 0 \quad \rightarrow \quad x^2 = ½v \pm \sqrt{¼v^2 + u^2}. \tag{12}$$

The minus sign for the radical is unacceptable since x^2 should be non-negative. Therefore,

$$x = \pm(\sqrt{¼v^2 + u^2} + ½v)^{½}. \tag{13}$$

In similar fashion, substituting $x = u/y$ in the equation $x^2 - y^2 = v$ yields

$$(u/y)^2 - y^2 = v \quad \rightarrow \quad y^4 + vy^2 - u^2 = 0 \quad \rightarrow \quad y^2 = -½v \pm \sqrt{¼v^2 + u^2}$$
$$\rightarrow \quad y = \pm(\sqrt{¼v^2 + u^2} - ½v)^{½}. \tag{14}$$

The partial derivatives of x and y with respect to u and v are readily found to be

$$\partial x/\partial u = \pm½(|u|/\sqrt{¼v^2 + u^2})(\sqrt{¼v^2 + u^2} + ½v)^{-½}, \tag{15}$$

$$\partial x/\partial v = \pm½(¼|v|/\sqrt{¼v^2 + u^2} + ½)(\sqrt{¼v^2 + u^2} + ½v)^{-½}, \tag{16}$$

$$\partial y/\partial u = \pm½(|u|/\sqrt{¼v^2 + u^2})(\sqrt{¼v^2 + u^2} - ½v)^{-½}. \tag{17}$$

$$\partial y/\partial v = \pm½(¼\,|v|/\sqrt{¼v^2 + u^2} - ½)(\sqrt{¼v^2 + u^2} - ½v)^{-½}. \tag{18}$$

The $\pm$ signs in the above equations pertain to the relevant quadrant of the xy-plane. In the expression of the Jacobian, however, $(\partial x/\partial u)(\partial y/\partial v)$ and $(\partial y/\partial u)(\partial x/\partial v)$ will carry the same sign (i.e., both + or both −). The final result then confirms Eqs.(9) and (10), as follows:

$$\frac{\partial(x,y)}{\partial(u,v)} = \left|(\pm⅛)\left(\frac{½|v|}{¼v^2+u^2} - \frac{1}{\sqrt{¼v^2+u^2}}\right) - (\pm⅛)\left(\frac{1}{\sqrt{¼v^2+u^2}} + \frac{½|v|}{¼v^2+u^2}\right)\right| = \frac{1}{2\sqrt{v^2+4u^2}}. \tag{19}$$
