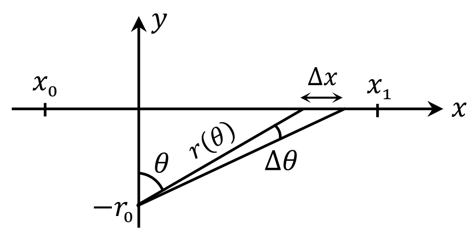
Problem 10) The composite function is $A(r_1, r_2, \cdots, r_N) + \lambda P(r_1, r_2, \cdots, r_N)$. Setting the partial derivative of the composite function with respect to r_n equal to zero, we will have

$$\partial_{r_n} A + \lambda \partial_{r_n} P = r_n \Delta\theta + \lambda \Delta\theta = 0 \quad \rightarrow \quad r_n = -\lambda. \tag{1}$$

Next, we enforce the constraint $P = P_0$. We find $P = \sum_n r_n \Delta\theta = -\lambda \sum_n \Delta\theta = -2\pi\lambda = P_0$, which yields $\lambda = -P_0/2\pi$. We will then have $r_1 = r_2 = \cdots = r_N = -\lambda = P_0/2\pi$. The curve which encloses the maximum area A for a given perimeter $P = P_0$ is a circle of radius $r = P_0/2\pi$.

Digression. While the enclosed area A is accurately represented by $\int_{\theta=0}^{2\pi} \tfrac{1}{2} r^2(\theta) \mathrm{d}\theta$, the expression used in this problem for the perimeter P, namely, $\int_{\theta=0}^{2\pi} r(\theta) \mathrm{d}\theta$, is reasonably accurate provided that the closed contour does not deviate too far from a circle. To appreciate this, note in the diagram on the right-hand side that the integral of $\tfrac{1}{2} r^2(\theta)$ yields the expected area of a triangle, whereas the integral of $r(\theta)$ does *not* yield the expected length of a line segment; that is,



$$\int_{\theta_0}^{\theta_1} \tfrac{1}{2} r^2(\theta) \mathrm{d}\theta = \tfrac{1}{2} \int_{\theta_0}^{\theta_1} (r_0/\cos\theta)^2 \mathrm{d}\theta = \tfrac{1}{2} r_0^2 \int_{\theta_0}^{\theta_1} (1 + \tan^2\theta) \mathrm{d}\theta$$
$$= \tfrac{1}{2} r_0^2 (\tan\theta_1 - \tan\theta_0) = \tfrac{1}{2} r_0 (x_1 - x_0). \tag{2}$$

$$\int_{\theta_0}^{\theta_1} r(\theta) \mathrm{d}\theta = \int_{\theta_0}^{\theta_1} (r_0/\cos\theta) \mathrm{d}\theta = r_0 \int_{\theta_0}^{\theta_1} \frac{1 + \tan^2(\tfrac{1}{2}\theta)}{1 - \tan^2(\tfrac{1}{2}\theta)} \mathrm{d}\theta = 2r_0 \int_{\tan(\frac{1}{2}\theta_0)}^{\tan(\frac{1}{2}\theta_1)} \frac{\mathrm{d}\zeta}{1 - \zeta^2}$$
$$= r_0 \int_{\tan(\frac{1}{2}\theta_0)}^{\tan(\frac{1}{2}\theta_1)} \left(\frac{1}{1+\zeta} + \frac{1}{1-\zeta} \right) \mathrm{d}\zeta = r_0 [\ln(1+\zeta) - \ln(1-\zeta)]_{\zeta=\tan(\frac{1}{2}\theta_0)}^{\tan(\frac{1}{2}\theta_1)}$$
$$= r_0 \ln\left[\frac{1+\tan(\tfrac{1}{2}\theta_1)}{1-\tan(\tfrac{1}{2}\theta_1)}\right] - r_0 \ln\left[\frac{1+\tan(\tfrac{1}{2}\theta_0)}{1-\tan(\tfrac{1}{2}\theta_0)}\right]. \tag{3}$$

When $\tan(\tfrac{1}{2}\theta)$ is sufficiently small, we will have $\ln[1 \pm \tan(\tfrac{1}{2}\theta)] \cong \pm\tan(\tfrac{1}{2}\theta) \cong \pm\tfrac{1}{2}\theta$, in which case the expression on the right-hand side of Eq.(3) reduces to $r_0(\theta_1 - \theta_0) \cong x_1 - x_0$. In general, however, $\int_{\theta=0}^{2\pi} r(\theta) \mathrm{d}\theta$ is *not* equal to $x_1 - x_0$.

A more accurate formula for the perimeter is $P = \sum_{n=1}^{N} \sqrt{r_{n-1}^2 + r_n^2 - 2r_{n-1} r_n \cos(\Delta\theta)}$. One could also express the area as $A = \tfrac{1}{2} \sum_{n=1}^{N} r_{n-1} r_n \sin(\Delta\theta)$. In these equations, $r_0 = r_N$ and $r_{N+1} = r_1$. Equating to zero the derivative with respect to r_n of the composite function $A + \lambda P$, we arrive at

$$\partial_{r_n} A + \lambda \partial_{r_n} P = \tfrac{1}{2}(r_{n-1} + r_{n+1}) \sin(\Delta\theta) + \frac{\lambda[r_n - r_{n-1}\cos(\Delta\theta)]}{[r_{n-1}^2 + r_n^2 - 2r_{n-1} r_n \cos(\Delta\theta)]^{\frac{1}{2}}}$$
$$+ \frac{\lambda[r_n - r_{n+1}\cos(\Delta\theta)]}{[r_n^2 + r_{n+1}^2 - 2r_n r_{n+1} \cos(\Delta\theta)]^{\frac{1}{2}}} = 0, \qquad (n = 1, 2, \cdots, N). \tag{4}$$

Here, we have N coupled nonlinear algebraic equations that must be solved for $r_1, r_2, \cdots, r_N$ as functions of λ. This is not an easy problem; however, it is not difficult to observe that a legitimate solution to the above set of equations is $r_1 = r_2 = \cdots = r_N = r$, in which case,

$$r\sin(\Delta\theta) + 2\lambda\sin(\Delta\theta/2) = 0 \quad \rightarrow \quad r = -\lambda/\cos(\Delta\theta/2). \tag{5}$$

Forcing the constraint now yields

$$P = \sum_{n=1}^{N}\sqrt{r_{n-1}^2 + r_n^2 - 2r_{n-1}r_n\cos(\Delta\theta)} = 2Nr\sin(\Delta\theta/2) = -2N\lambda\tan(\Delta\theta/2) = P_0. \tag{6}$$

Recalling that $\Delta\theta$ is chosen to be a small angle, we will have $2N\tan(\Delta\theta/2) \cong N\Delta\theta = 2\pi$, and, therefore, $\lambda = -P_0/2\pi$. Upon substituting into Eq.(5) and noting that $\cos(\Delta\theta/2) \cong 1$, we finally arrive at $r_1 = r_2 = \cdots = r_N = r = P_0/2\pi$. The maximized area is subsequently found to be

$$A = ½\sum_{n=1}^{N} r_{n-1}r_n\sin(\Delta\theta) \cong ½r^2\sum_{n=1}^{N}\Delta\theta = \pi r^2. \tag{7}$$