Problem 3) a) The particular solution is zero because $\delta(t)=0$ for $t>0$. The homogeneous solution for $t>0$ is taken to be $z(t)=Ae^{\eta t}$. Substitution into the differential equation yields

$$A\eta^2 e^{\eta t}+\gamma A\eta e^{\eta t}+\omega_0^2 Ae^{\eta t}=0 \;\rightarrow\; \eta^2+\gamma\eta+\omega_0^2=0 \;\rightarrow\; \eta_{1,2}=-\tfrac{1}{2}\gamma\pm \mathrm{i}\sqrt{\omega_0^2-\tfrac{1}{4}\gamma^2}. \quad (1)$$

The solution for $t<0$ is $z(t)=0$; for $t>0$ it is $z(t)=Ae^{\eta_1 t}+Be^{\eta_2 t}$. The solution must be continuous at $t=0$, which necessitates that $A+B=0$. As for the derivative of $z(t)$, it must jump at $t=0$, so that the second derivative $z''(t)$ becomes a delta-function at $t=0$ to match the delta-function appearing on the right-hand side of the differential equation. The jump of $z'(t)$ at $t=0$ is equal to $A\eta_1+B\eta_2$, which makes $z''(t)$ on the left-hand side equal to $(A\eta_1+B\eta_2)\delta(t)$. Equating this to the right-hand side now yields $A\eta_1+B\eta_2=f_0/m$. Therefore,

$$A=-B=f_0/[m(\eta_1-\eta_2)]=f_0/(2\mathrm{i}m\sqrt{\omega_0^2-\tfrac{1}{4}\gamma^2}). \quad (2)$$

Substituting for A and B into the solution for $t>0$, and introducing $\text{step}(t)$ to incorporate the solution for $t<0$, now yields the complete solution of the differential equation, as follows:

$$z(t)=f_0\text{step}(t)e^{-\frac{1}{2}\gamma t}\,[\exp(\mathrm{i}\sqrt{\omega_0^2-\tfrac{1}{4}\gamma^2}\,t)-\exp(-\mathrm{i}\sqrt{\omega_0^2-\tfrac{1}{4}\gamma^2}\,t)]/(2\mathrm{i}m\sqrt{\omega_0^2-\tfrac{1}{4}\gamma^2}). \quad (3)$$

This solution is readily simplified as

$$z(t)=(f_0/m)\text{step}(t)\,e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\omega_0^2-\tfrac{1}{4}\gamma^2}\,t)/\sqrt{\omega_0^2-\tfrac{1}{4}\gamma^2}. \quad (4)$$

b) Using $z(t)=\int_{-\infty}^{\infty}Z(s)e^{\mathrm{i}2\pi st}\mathrm{d}s$ and $\delta(t)=\int_{-\infty}^{\infty}e^{\mathrm{i}2\pi st}\mathrm{d}s$, we take the equation of motion to the Fourier domain, arriving at

$$(\mathrm{i}2\pi s)^2Z(s)+\gamma(\mathrm{i}2\pi s)Z(s)+\omega_0^2Z(s)=f_0/m. \quad (5)$$

The above equation is readily solved for $Z(s)$, yielding

$$Z(s)=\frac{f_0/m}{-4\pi^2s^2+\mathrm{i}2\pi\gamma s+\omega_0^2}=-\left(\frac{f_0}{4\pi^2m}\right)\frac{1}{s^2-\mathrm{i}(\gamma/2\pi)s-(\omega_0/2\pi)^2}=-\left(\frac{f_0}{4\pi^2m}\right)\frac{1}{(s-s_1)(s-s_2)}, \quad (6)$$

where $s_{1,2}=\left(\pm\sqrt{\omega_0^2-\tfrac{1}{4}\gamma^2}+\tfrac{1}{2}\mathrm{i}\gamma\right)/2\pi$. The inverse Fourier transform of $Z(s)$ is given by

$$z(t)=\int_{-\infty}^{\infty}Z(s)e^{\mathrm{i}2\pi st}\mathrm{d}s=-\left(\frac{f_0}{4\pi^2m}\right)\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}2\pi st}}{(s-s_1)(s-s_2)}\mathrm{d}s. \quad (7)$$

Both poles s_1 and s_2 of the integrand are in the upper-half of the complex plane. For $t<0$, the integration contour must be closed with a large semi-circle in the lower half-plane, where Jordan's lemma applies (because $e^{\mathrm{i}2\pi st}\rightarrow 0$ on the large semi-circle in the lower half-plane). Consequently, $z(t)=0$ for $t<0$. For $t>0$, the integration contour must be closed with a large semi-circle in the upper half-plane, where $e^{\mathrm{i}2\pi st}\rightarrow 0$ and Jordan's lemma applies once again. The value of the integral will then be $\mathrm{i}2\pi$ times the sum of the residues at s_1 and s_2; that is,

$$z(t)=-\left(\frac{f_0}{4\pi^2m}\right)(\mathrm{i}2\pi)\left(\frac{e^{\mathrm{i}2\pi s_1t}}{s_1-s_2}+\frac{e^{\mathrm{i}2\pi s_2t}}{s_2-s_1}\right)=-\left(\frac{\mathrm{i}f_0}{2\pi m}\right)\left(\frac{e^{\mathrm{i}2\pi s_1t}-e^{\mathrm{i}2\pi s_2t}}{s_1-s_2}\right). \quad (8)$$

Further simplification of this result followed by the introduction of $\text{step}(t)$ in order to incorporate the solution for $t<0$, finally yields

$$z(t)=(f_0/m)\text{step}(t)e^{-\frac{1}{2}\gamma t}\sin\left(\sqrt{\omega_0^2-\tfrac{1}{4}\gamma^2}\,t\right)/\sqrt{\omega_0^2-\tfrac{1}{4}\gamma^2}. \quad (9)$$

This is the same solution as obtained in part (a), Eq.(4), using the conventional method.

Digression 1. To confirm that the solution obtained for $z(t)$ does in fact satisfy the original differential equation, we begin by evaluating the first and second derivatives of $z(t)$, namely,

$$z(t) = (f_0/m\sqrt{\cdots})\,\text{step}(t)e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\cdots}\,t). \tag{10}$$

$$z'(t) = (f_0/m\sqrt{\cdots})\big[\delta(t)e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\cdots}\,t) - \tfrac{1}{2}\gamma\,\text{step}(t)e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\cdots}\,t)$$
$$+\sqrt{\cdots}\,\text{step}(t)e^{-\frac{1}{2}\gamma t}\cos(\sqrt{\cdots}\,t)\big]. \tag{11}$$

$$z''(t) = (f_0/m\sqrt{\cdots})\big[\delta'(t)e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\cdots}\,t) - \tfrac{1}{2}\gamma\delta(t)e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\cdots}\,t) + \sqrt{\cdots}\,\delta(t)e^{-\frac{1}{2}\gamma t}\cos(\sqrt{\cdots}\,t)$$
$$-\tfrac{1}{2}\gamma\,\delta(t)e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\cdots}\,t) + \tfrac{1}{4}\gamma^2\,\text{step}(t)e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\cdots}\,t) - \tfrac{1}{2}\sqrt{\cdots}\,\gamma\,\text{step}(t)e^{-\frac{1}{2}\gamma t}\cos(\sqrt{\cdots}\,t)$$
$$+\sqrt{\cdots}\;\delta(t)e^{-\frac{1}{2}\gamma t}\cos(\sqrt{\cdots}\,t) - \tfrac{1}{2}\gamma\sqrt{\cdots}\,\text{step}(t)e^{-\frac{1}{2}\gamma t}\cos(\sqrt{\cdots}\,t) - (\sqrt{\cdots})^2\,\text{step}(t)e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\cdots}\,t)\big]. \tag{12}$$

The governing differential equation may thus be written as

$$z''(t) + \gamma z'(t) + \omega_0^2 z(t) = (f_0/m\sqrt{\cdots})\big[e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\cdots}\,t)\,\delta'(t) + 2\sqrt{\cdots}\,e^{-\frac{1}{2}\gamma t}\cos(\sqrt{\cdots}\,t)\,\delta(t)\big]. \tag{13}$$

Recalling that the sifting properties of $\delta(t)$ and $\delta'(t)$ acting on the function $f(t)g(t)$ are given by

$$\int_{-\infty}^{\infty} f(t)g(t)\delta(t)\mathrm{d}t = f(0)g(0) \quad \text{and} \quad \int_{-\infty}^{\infty} f(t)g(t)\delta'(t)\mathrm{d}t = -[f'(0)g(0) + f(0)g'(0)],$$

we conclude that $g(t)\delta'(t) = g(0)\delta'(t) - g'(0)\delta(t)$. Therefore, the first term on the right-hand side of Eq.(13) becomes

$$e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\cdots}\,t)\,\delta'(t) = e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\cdots}\,t)\Big|_{t=0}\delta'(t)$$
$$-\big[-\tfrac{1}{2}\gamma e^{-\frac{1}{2}\gamma t}\sin(\sqrt{\cdots}\,t) + \sqrt{\cdots}\,e^{-\frac{1}{2}\gamma t}\cos(\sqrt{\cdots}\,t)\big]_{t=0}\delta(t) = -\sqrt{\cdots}\,\delta(t). \tag{14}$$

Substitution from Eq.(14) into Eq.(13) finally yields the original differential equation, as follows:

$$z''(t) + \gamma z'(t) + \omega_0^2 z(t) = (f_0/m\sqrt{\cdots})\big[-\sqrt{\cdots}\,\delta(t) + 2\sqrt{\cdots}\,\delta(t)\big] = (f_0/m)\delta(t). \tag{15}$$

Digression 2. A general approach to solving the mass-and-spring problem when $\delta(t)$ and/or its derivative $\delta'(t)$ appear on the right-hand side of the equation of motion is to proceed as follows:

$$z(t) = \text{step}(t)(Ae^{\eta_1 t} + Be^{\eta_2 t}). \tag{16}$$

$$z'(t) = \delta(t)(Ae^{\eta_1 t} + Be^{\eta_2 t}) + \text{step}(t)(A\eta_1 e^{\eta_1 t} + B\eta_2 e^{\eta_2 t}). \tag{17}$$

$$z''(t) = \delta'(t)(Ae^{\eta_1 t} + Be^{\eta_2 t}) + 2\delta(t)(A\eta_1 e^{\eta_1 t} + B\eta_2 e^{\eta_2 t}) + \text{step}(t)(A\eta_1^2 e^{\eta_1 t} + B\eta_2^2 e^{\eta_2 t})$$

invoke $g(t)\delta'(t) = g(0)\delta'(t) - g'(0)\delta(t)$; also, $g(t)\delta(t) = g(0)\delta(t)$

$$= \delta'(t)(A + B) + \delta(t)(A\eta_1 + B\eta_2) + \text{step}(t)(A\eta_1^2 e^{\eta_1 t} + B\eta_2^2 e^{\eta_2 t}). \tag{18}$$

Substitution into the equation of motion yields

$$z''(t) + \gamma z'(t) + \omega_0^2 z(t) = (A + B)[\delta'(t) + \gamma\delta(t)] + (A\eta_1 + B\eta_2)\delta(t)$$
$$+\text{step}(t)[A\cancelto{0}{(\eta_1^2 + \gamma\eta_1 + \omega_0^2)}e^{\eta_1 t} + B\cancelto{0}{(\eta_2^2 + \gamma\eta_2 + \omega_0^2)}e^{\eta_2 t}]. \tag{19}$$

When the right-hand side of the differential equation is $(f_0/m)\delta(t)$, we must set $A + B = 0$ to eliminate $\delta'(t)$ from the left-hand side, then set $A\eta_1 + B\eta_2 = A(\eta_1 - \eta_2) = f_0/m$ in order to satisfy the equation. In contrast, if the term on the right-hand side happens to be $(f_0/m)\delta'(t)$, then we set $A + B = f_0/m$ and $(A + B)\gamma + A\eta_1 + B\eta_2 = 0$ to once again satisfy the equation of motion. If it so happens that the right-hand side of the governing equation contains the second derivative $\delta''(t)$ of $\delta(t)$, we must consider adding the new term $C\delta(t)$ to the postulated solution $z(t)$ appearing in Eq.(16), where C is another constant (similar to A and B) that must be determined by balancing the right- and left-hand sides of the differential equation, and so on.
