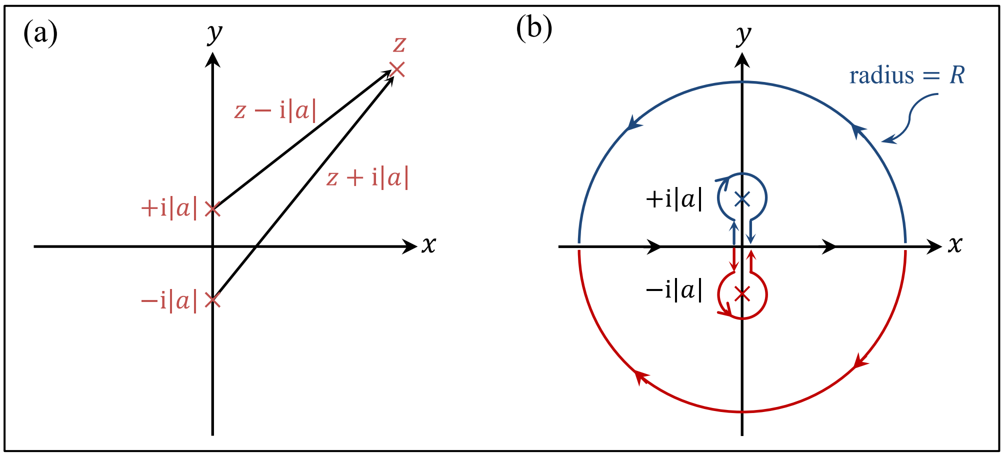
Problem 2) If $a = 0$, $f(x)$ will be zero everywhere and, therefore, its Fourier transform will be $F(s) = 0$. For nonzero values of a, the function $f(z) = 2a/(z^2 + a^2)$ will have two first-order poles at $z_{1,2} = \pm \mathrm{i}a$ in the complex z-plane. We will have

$$|f(z)| = \frac{2|a|}{|z-\mathrm{i}a|\,|z+\mathrm{i}a|}.$$

In the limit when $|z| \to \infty$, both $|z - \mathrm{i}a|$ and $|z + \mathrm{i}a|$ approach infinity, as can be inferred from figure (a) below. Consequently, $lim_{|z|\to\infty}|f(z)| = 0$ and the condition for Jordan's lemma is satisfied.



The Fourier integral $F(s) = \int_{-\infty}^{\infty} f(x)e^{-\mathrm{i}2\pi sx}\mathrm{d}x$ should be evaluated in the upper-half plane when $s < 0$, and in the lower half-plane when $s > 0$. This is because, at $z = Re^{\mathrm{i}\varphi}$ on the circle of large radius R in figure (b), the Fourier kernel $|e^{-\mathrm{i}2\pi sz}| = |e^{-\mathrm{i}2\pi sR\cos\varphi}e^{2\pi sR\sin\varphi}| = e^{2\pi sR\sin\varphi}$ goes to zero on the upper semi-circle when $s < 0$, and on the lower semi-circle when $s > 0$. The only pole in the upper half-plane is at $z_1 = \mathrm{i}|a|$ (first-order), where the residue of the integrand $2ae^{-\mathrm{i}2\pi sz}/[(z - \mathrm{i}|a|)(z + \mathrm{i}|a|)]$ is found to be

$$2ae^{-\mathrm{i}2\pi s(\mathrm{i}|a|)}/(\mathrm{i}|a| + \mathrm{i}|a|) = -\mathrm{i}\ \mathrm{sign}(a)e^{2\pi|a|s} \qquad (s < 0).$$

In the lower half-plane, the only pole is at $z_2 = -\mathrm{i}|a|$ (also first order). Here the residue of the integrand is

$$2ae^{-\mathrm{i}2\pi s(-\mathrm{i}|a|)}/(-\mathrm{i}|a| - \mathrm{i}|a|) = \mathrm{i}\ \mathrm{sign}(a)e^{-2\pi|a|s} \qquad (s > 0).$$

Considering that the contribution of the upper-half-plane residue to the overall integral must be multiplied by $\mathrm{i}2\pi$ (counterclockwise contour around $z_1 = \mathrm{i}|a|$), and that of the lower-half-plane residue multiplied by $-\mathrm{i}2\pi$ (clockwise contour around $z_2 = -\mathrm{i}|a|$), we finally arrive at

$$\boxed{F(s) = 2\pi\ \mathrm{sign}(a)e^{-2\pi|as|}.}$$

Note that the area under $F(s)$ is $\int_{-\infty}^{\infty} 2\pi\ \mathrm{sign}(a)e^{-2\pi|as|}\mathrm{d}s = 4\pi\ \mathrm{sign}(a)\int_0^{\infty} e^{-2\pi|a|s}\mathrm{d}s = 4\pi\ \mathrm{sign}(a)/(2\pi|a|) = 2/a$, in agreement with the value of the function $f(x)$ at $x = 0$.

Digression: For $s = 0$, the integration contour may be closed in either the upper half-plane or the lower-half plane, yielding $F(0) = 2\pi\,\mathrm{sign}(a)$. It is also possible in this case to directly evaluate the Fourier integral; that is,

change of variable: $x = ay$; limits of the integral switch sign for negative a

$$F(0) = \int_{-\infty}^{\infty} f(x)\mathrm{d}x = \int_{-\infty}^{\infty} \frac{2a\mathrm{d}x}{x^2+a^2} = 2a^2 \int_{\mp\infty}^{\pm\infty} \frac{\mathrm{d}y}{(ay)^2+a^2} = 2\int_{\mp\infty}^{\pm\infty} \frac{\mathrm{d}y}{y^2+1} = 2\,\mathrm{sign}(a) \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{1+y^2}$$

change of variable: $y = \tan\theta$

$$= 2\,\mathrm{sign}(a) \int_{-\pi/2}^{\pi/2} \frac{1+\tan^2\theta}{1+\tan^2\theta}\mathrm{d}\theta = 2\,\mathrm{sign}(a) \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta = 2\pi\,\mathrm{sign}(a).$$
