Opti 403A/503A **Final Exam** (5/6/2024) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

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6 pts **Problem 1**) a) Evaluate $\int_{-L/2}^{L/2} e^{\mathrm{i}2\pi sx}\mathrm{d}x$, where L is an arbitrary positive number. Argue that, in the limit when $L \to \infty$, the integral approaches a delta-function, namely, $\int_{-\infty}^{\infty} e^{\mathrm{i}2\pi sx}\mathrm{d}x = \delta(s)$.

6 pts b) The standard expression of the Fourier transform of the product function $h(x) = f(x)g(x)$ is $H(s) = \int_{-\infty}^{\infty} f(x)g(x)e^{-\mathrm{i}2\pi sx}\mathrm{d}x$. Let the Fourier transforms of the functions $f(x)$ and $g(x)$ be $F(s)$ and $G(s)$, respectively. Substitute the inverse Fourier integrals for $f(x)$ and $g(x)$ into the above standard expression of $H(s)$ to arrive at a triple integral involving three independent variables s', s'', and x.

6 pts c) Rearrange the triple integral obtained in (b), then use the result of part (a) to reduce the triple integral to a double integral over the independent variables s' and s''.

6 pts d) Invoke the sifting property of the delta-function to reduce the double integral obtained in (c) to a single integral over the remaining variable s'. As expected, your final result should be a convolution integral, namely, $H(s) = \int_{-\infty}^{\infty} F(s')G(s-s')\mathrm{d}s'$.

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12 pts **Problem 2**) Use the methods of complex-plane integration to find the Fourier transform $F(s)$ of the function $f(x) = 2a/(x^2 + a^2)$, where a is a real-valued but otherwise arbitrary constant.

Hint: You should consider the possibility that the parameter a could be less than, equal to, or greater than zero. Note that the Fourier integrals for $s > 0$ and $s < 0$ should be evaluated on different contours in the complex z-plane.

Problem 3) In the mass-and-spring model of a harmonic oscillator, the differential equation $z''(t) + \gamma z'(t) + \omega_0^2 z(t) = m^{-1}f(t)$ has been solved in two different ways (the conventional way and the Fourier transform way), and the results are found to be identical. In this problem, you are asked to solve the problem using both methods in the special case where $f(t) = f_0\delta(t)$ (i.e., impulsive excitation) and $\gamma < 2\omega_0^2$ (i.e., underdamped system).

8pts a) Set $z(t) = 0$ for $t < 0$. Recognize that the particular solution for $t > 0$ also equals zero (because, for the impulsive excitation, $\delta(t) = 0$ for $t > 0$). Write down the homogeneous solution for $t > 0$. Proceed to calculate the initially unknown parameters of this solution (this involves choosing the initial conditions at $t = 0^+$) to arrive at a complete solution.

6 pts b) Apply the Fourier transformation method to the differential equation in order to find the Fourier transform $Z(s)$ of the solution. Proceed to use the complex-plane integration techniques to determine $z(t)$, which is the inverse Fourier transform of $Z(s)$.
