Problem 4) a) $f(x) = x/\sum_{m=0}^{2} a_m x^m = \sum_{n=0}^{\infty} c_n x^n$ $\boxed{n+m=k}$

$$\rightarrow \quad x = \sum_{n=0}^{\infty} c_n x^n \sum_{m=0}^{2} a_m x^m = \sum_{n=0}^{\infty} \sum_{m=0}^{2} a_m c_n x^{n+m} = \sum_{k=0}^{\infty} \left(\sum_{m=0}^{2} a_m c_{k-m}\right) x^k.$$

For the first couple of terms, where $k = 0$ and $k = 1$, the above equation yields

$$k = 0: \quad a_0 c_0 = 0 \quad \rightarrow \quad c_0 = 0,$$

$$k = 1: \quad a_0 c_1 + a_1 c_0 = 1 \quad \rightarrow \quad c_1 = 1 + 3c_0 = 1.$$

As for the remaining terms with $k \geq 2$, we find the following recursion relation:

$$k \geq 2: \quad a_0 c_k + a_1 c_{k-1} + a_2 c_{k-2} = 0 \quad \rightarrow \quad c_k = 3c_{k-1} - c_{k-2}.$$

b) The coefficients c_k for $k = 2$ up to $k = 7$ are now found to be

$$c_2 = 3c_1 - c_0 = 3,$$

$$c_3 = 3c_2 - c_1 = 8,$$

$$c_4 = 3c_3 - c_2 = 21,$$

$$c_5 = 3c_4 - c_3 = 55,$$

$$c_6 = 3c_5 - c_4 = 144,$$

$$c_7 = 3c_6 - c_5 = 377.$$

The Taylor series expansion of $f(x)$ up to and including the seventh order term is thus given by

$$f(x) = x + 3x^2 + 8x^3 + 21x^4 + 55x^5 + 144x^6 + 377x^7 + \cdots.$$

c) The values of $f(x)$ and its Taylor series expansion (up to the 7^{th} order) at $x = 0.1$ are given by

$$f(0.1) = \frac{0.1}{0.01 - 0.3 + 1} = 0.14084507 \cdots$$

$$f(0.1) = 0.1 + 0.03 + 0.008 + 0.0021 + 0.00055 + 0.000144 + 0.0000377 + \cdots$$

$$\cong 0.1408317$$

The values of $f(x)$ and its Taylor series expansion (up to the 7^{th} order) at $x = -0.1$ are given by

$$f(-0.1) = \frac{-0.1}{0.01 + 0.3 + 1} = -0.076335878 \cdots$$

$$f(-0.1) = -0.1 + 0.03 - 0.008 + 0.0021 - 0.00055 + 0.000144 - 0.0000377 + \cdots$$

$$\cong -0.0763437.$$

Digression: Since the closest singular point of $f(z) = z/(z^2 - 3z + 1)$ to $z_0 = x_0 = 0$ in the complex z-plane is $z_2 = ½(3 - \sqrt{5}) + 0i = 0.381966 \cdots + 0i$, the domain of convergence of our Taylor series can be shown to be a circle of radius $0.381966 \cdots$ centered at $z_0 = 0$. Therefore, on the real axis, the Taylor series expansion is convergent for $|x| < 0.381966 \cdots$.