Problem 3) a) $\sum_{n=1}^{\infty}(1/n^4) = \sum_{n=1}^{\infty} 1/(2n)^4 + \sum_{n=1}^{\infty} 1/(2n-1)^4 = \frac{1}{16}\sum_{n=1}^{\infty}(1/n^4) + (\pi^4/96)$

$$\rightarrow \quad \frac{15}{16}\sum_{n=1}^{\infty}(1/n^4) = \pi^4/96 \quad \rightarrow \quad \sum_{n=1}^{\infty}(1/n^4) = \pi^4/90.$$

b) $\int_0^1 [(\ln x)^3/(1-x)]\mathrm{d}x = \sum_{n=0}^{\infty}\int_0^1 x^n(\ln x)^3 \mathrm{d}x$

$$\boxed{\mathrm{d}(\ln x)^3/\mathrm{d}x = (3/x)(\ln x)^2}$$

$$= \sum_{n=0}^{\infty}\left[\cancelto{0}{\frac{x^{n+1}}{n+1}(\ln x)^3\bigg|_{x=0}^{1}} - \int_0^1 \frac{x^{n+1}}{n+1}\cdot\frac{3}{x}(\ln x)^2 \mathrm{d}x\right] = -\sum_{n=0}^{\infty}\frac{3}{n+1}\int_0^1 x^n(\ln x)^2 \mathrm{d}x$$

$$\boxed{\mathrm{d}(\ln x)^2/\mathrm{d}x = (2/x)\ln x}$$

$$= -\sum_{n=0}^{\infty}\left[\cancelto{0}{\frac{3x^{n+1}}{(n+1)^2}(\ln x)^2\bigg|_{x=0}^{1}} - \frac{3}{n+1}\int_0^1 \frac{x^{n+1}}{n+1}\cdot\frac{2}{x}\ln x \,\mathrm{d}x\right] = \sum_{n=0}^{\infty}\frac{6}{(n+1)^2}\int_0^1 x^n \ln x \,\mathrm{d}x$$

$$= \sum_{n=0}^{\infty}\left[\cancelto{0}{\frac{6x^{n+1}}{(n+1)^3}\ln x\bigg|_{x=0}^{1}} - \frac{6}{(n+1)^2}\int_0^1 \frac{x^{n+1}}{n+1}\cdot\frac{1}{x}\mathrm{d}x\right] = -\sum_{n=0}^{\infty}\frac{6}{(n+1)^3}\int_0^1 x^n \mathrm{d}x$$

$$= -\sum_{n=0}^{\infty}\frac{6x^{n+1}}{(n+1)^4}\bigg|_{x=0}^{1} = -6\sum_{n=1}^{\infty}(1/n^4) = -\pi^4/15.$$

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Digression: To prove that $lim_{(x\to 0^+)}\, x(\ln x)^\alpha = 0$, let $x = e^{-k}$, where k is a large positive integer. You will have $x(\ln x)^\alpha = e^{-k}(-k)^\alpha = e^{\mathrm{i}\pi\alpha}k^\alpha/e^k$. Given that, with an increasing k, the value of e^k grows faster than that of k^α, you will find that $k^\alpha/e^k \rightarrow 0$ in the limit when $k \rightarrow \infty$.