Opti 403A/503A **Midterm Exam** (3/13/2024) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

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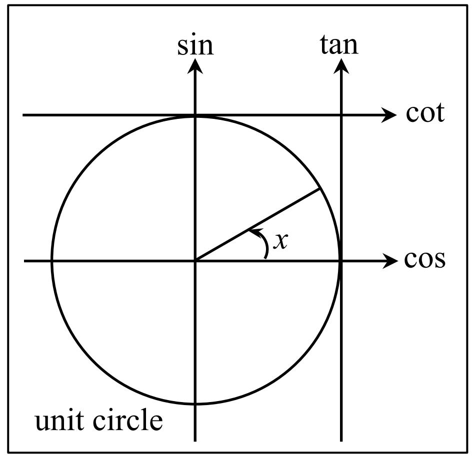
5 pts **Problem 1**) Trigonometric identities appear frequently in geometric optics — not to mention numerous other areas of science and engineering. (a) Use the circle diagram (depicted below) to prove the following identities:

$\sin(\pi - x) = \sin x$; $\cos(\pi - x) = -\cos x$;

$\sin(x - ½\pi) = -\cos x$; $\cos(½\pi + x) = -\sin x$;

$\cos(\pi + x) = -\cos x$; $\tan(\pi - x) = -\tan(x)$.



5 pts b) Invoke Euler's identity, $e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$, along with the fundamental property of the exponential function, $e^{\mathrm{i}(x+y)} = e^{\mathrm{i}x}e^{\mathrm{i}y}$, to prove the identities listed below.

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$;

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$;

$\sin(2x) = 2\sin x \cos x$

$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$;

$\tan(x \pm y) = (\tan x \pm \tan y)/(1 \mp \tan x \tan y)$.

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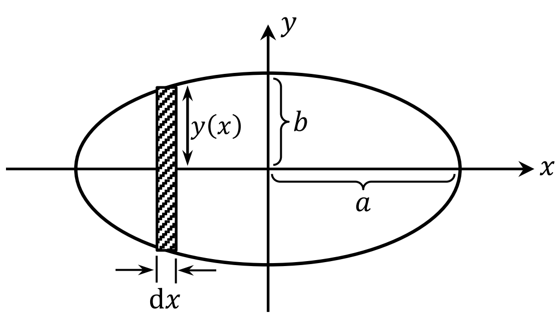
7 pts **Problem 2**) An ellipse in the xy-plane is defined by the equation $(x/a)^2 + (y/b)^2 = 1$, where a and b are the lengths of the ellipse's semi-axes in the x and y directions, respectively. Without resorting to curvilinear coordinates to simplify the calculation, use direct integration to determine the area of the ellipse, namely, $A = \iint_{\text{ellipse's surface}} \mathrm{d}x\mathrm{d}y$.



Hint: The change of variable $x \to a\sin\theta$ could be helpful. You may also want to invoke the trig identity $\cos^2\theta = ½[1 + \cos(2\theta)]$.

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3 pts **Problem 3**) a) The sum of the inverses of the 4th power of odd integers is known to be $\sum_{n=1}^{\infty} 1/(2n-1)^4 = \pi^4/96$. Use this result to show that $\sum_{n=1}^{\infty}(1/n^4) = \pi^4/90$.

5 pts b) Invoke the result obtained in (a) to show that $\int_0^1 [(\ln x)^3/(1-x)]\mathrm{d}x = -\pi^4/15$. Note that the integrand is well-behaved in the vicinity of $x = 1$, since $\ln x \to (x-1)$ as x approaches 1.

Hint: Substitute the geometric series $\sum_{n=0}^{\infty} x^n$ for $1/(1-x)$; this is a valid substitution, considering that the domain of the integral is $0 \leq x < 1$. Use the method of integration by parts to evaluate the resulting integrals. Recall that $lim_{(x\to 0^+)}\, x(\ln x)^\alpha = 0$ for all values of α.

Problem 4) Consider the function $f(x) = x/(x^2 - 3x + 1)$, which is defined over the entire x-axis except at the two singular points $x_{1,2} = ½(3 \pm \sqrt{5})$, where the denominator of the function vanishes.

4 pts a) Use the method that was introduced in the class for finding Bernoulli's numbers to determine the Taylor series expansion of $f(x)$ around the point $x_0 = 0$; that is, $f(x) = \sum_{n=0}^{\infty} c_n x^n$.

4 pts b) Using the recursion formula obtained in part (a) for the Taylor series coefficients c_n, evaluate $c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7$.

2 pts c) Show that at $x = \pm 0.1$, the Taylor series expansion of $f(x)$ up to the seventh order provides good approximations to the actual values of $f(x = \pm 0.1)$.

Hint: You may want to start by writing $f(x) = x/\sum_{m=0}^{2} a_m x^m$, where $a_0 = 1$, $a_1 = -3$, and $a_2 = 1$.
