Problem 4) Since the integrand is even, one can extend the integral to the entire x-axis, from $-\infty$ to ∞. Invoking the identity $\cos(\zeta) = ½(e^{i\zeta} + e^{-i\zeta})$ and rearranging the integral, we arrive at

$$\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2(x^2+\beta^2)}\mathrm{d}x = ¼\int_{-\infty}^\infty \frac{\exp(\mathrm{i}ax)-\exp(\mathrm{i}bx)}{x^2(x^2+\beta^2)}\mathrm{d}x + ¼\int_{-\infty}^\infty \frac{\exp(-\mathrm{i}ax)-\exp(-\mathrm{i}bx)}{x^2(x^2+\beta^2)}\mathrm{d}x. \qquad (1)$$

For the first and second integrals appearing on the right-hand side of Eq.(1), the complex-plane integration contours are shown in figures (a) and (b), respectively. This is because, in accordance with Jordan's lemma in each case, the integrals on the large semi-circles vanish. Below, we evaluate the residues at the 2nd-order pole $z_0 = 0$ and the 1st-order poles $z_{1,2} = \pm\mathrm{i}\beta$.

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\exp(\pm\mathrm{i}az)-\exp(\pm\mathrm{i}bz)}{z^2+\beta^2}\right]_{z=0} = \frac{[\pm\mathrm{i}a\exp(\pm\mathrm{i}az)\mp\mathrm{i}b\exp(\pm\mathrm{i}bz)](z^2+\beta^2)-2z[\exp(\pm\mathrm{i}az)-\exp(\pm\mathrm{i}bz)]}{(z^2+\beta^2)^2}\Bigg|_{z=0}$$
$$= \pm\,\mathrm{i}(a-b)/\beta^2\,. \qquad (2)$$

Residue at $z_1 = \mathrm{i}\beta$: $$\frac{\exp(\mathrm{i}az)-\exp(\mathrm{i}bz)}{z^2(z+\mathrm{i}\beta)}\Bigg|_{z=z_1=\mathrm{i}\beta} = \frac{\exp(-a\beta)-\exp(-b\beta)}{-2\mathrm{i}\beta^3}. \qquad (3)$$

Residue at $z_2 = -\mathrm{i}\beta$: $$\frac{\exp(-\mathrm{i}az)-\exp(-\mathrm{i}bz)}{z^2(z-\mathrm{i}\beta)}\Bigg|_{z=z_2=-\mathrm{i}\beta} = \frac{\exp(-a\beta)-\exp(-b\beta)}{2\mathrm{i}\beta^3}. \qquad (4)$$

The first integral on the right-hand side of Eq.(1) is now given by $\mathrm{i}\pi$ times the corresponding residue at $z_0 = 0$, plus $\mathrm{i}2\pi$ times the residue at $z_1 = \mathrm{i}\beta$; that is,

$$\int_{-\infty}^\infty \frac{\exp(\mathrm{i}ax)-\exp(\mathrm{i}bx)}{x^2(x^2+\beta^2)}\mathrm{d}x = \mathrm{i}\pi\left[\frac{\mathrm{i}(a-b)}{\beta^2}\right] + \mathrm{i}2\pi\left[\frac{\exp(-a\beta)-\exp(-b\beta)}{-2\mathrm{i}\beta^3}\right]$$
$$= \frac{\pi[(b-a)\beta - \exp(-a\beta)+\exp(-b\beta)]}{\beta^3}. \qquad (5)$$

Note that, in compliance with figure (a), the small semi-circle around $z = z_0$ as well as the small circle around $z = z_1$ are traversed clockwise in connection with Eq.(5). As for the second integral on the right-hand side of Eq.(1), complying with figure (b) requires that both the small semi-circle around $z = z_0$ and the small circle around $z = z_2$ be traversed counterclockwise. The second integral is thus given by $-\mathrm{i}\pi$ times the corresponding residue at $z_0 = 0$, plus $-\mathrm{i}2\pi$ times the residue at $z_2 = -\mathrm{i}\beta$; that is,

$$\int_{-\infty}^\infty \frac{\exp(-\mathrm{i}ax)-\exp(-\mathrm{i}bx)}{x^2(x^2+\beta^2)}\mathrm{d}x = -\mathrm{i}\pi\left[-\frac{\mathrm{i}(a-b)}{\beta^2}\right] - \mathrm{i}2\pi\left[\frac{\exp(-a\beta)-\exp(-b\beta)}{2\mathrm{i}\beta^3}\right]$$
$$= \frac{\pi[(b-a)\beta - \exp(-a\beta)+\exp(-b\beta)]}{\beta^3}. \qquad (6)$$

Finally, substitution from Eqs.(5) and (6) into Eq.(1) yields

$$\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2(x^2+\beta^2)}\mathrm{d}x = \frac{\pi[(b-a)\beta + \exp(-b\beta)-\exp(-a\beta)]}{2\beta^3}. \qquad \text{(G\&R } \mathbf{3.784}\text{-8)} \qquad (7)$$