Problem 3) Invoking the convolution theorem of the Fourier transform theory, we write

$$\mathcal{F}\{\text{step}(x)f(x)\} = [\tfrac{1}{2}\delta(s) - \mathrm{i}/(2\pi s)] * F(s) = \tfrac{1}{2}\delta(s) * F(s) - (\mathrm{i}/2\pi s) * F(s)$$
$$= \tfrac{1}{2}F(s) - (\mathrm{i}/2\pi s) * [F_R(s) + \mathrm{i}F_I(s)]$$
$$= \tfrac{1}{2}F(s) - (\mathrm{i}/2\pi s) * F_R(s) + (1/2\pi s) * F_I(s).$$

Considering that, in the present problem, $\text{step}(x)f(x) = f(x)$, the left-hand side of the above equation must equal $F(s)$. Therefore,

$$\tfrac{1}{2}F(s) = \tfrac{1}{2}F_R(s) + \tfrac{1}{2}\mathrm{i}F_I(s) = (1/2\pi s) * F_I(s) - (\mathrm{i}/2\pi s) * F_R(s)$$
$$\rightarrow \quad F_R(s) = (1/\pi s) * F_I(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F_I(s')}{s-s'}\mathrm{d}s',$$

Also, $$F_I(s) = -(1/\pi s) * F_R(s) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{F_R(s')}{s-s'}\mathrm{d}s'.$$

Digression: The function $f(x)$ may be written as a superposition of $g(x) = \tfrac{1}{2}f(x) + \tfrac{1}{2}f^*(-x)$ and $h(x) = \tfrac{1}{2}f(x) - \tfrac{1}{2}f^*(-x)$; that is, $f(x) = g(x) + h(x)$. Next, we prove the following useful identity:

$$\mathcal{F}\{f^*(-x)\} = \int_{-\infty}^{0} f^*(-x)e^{-\mathrm{i}2\pi sx}\mathrm{d}x = \int_{0}^{\infty} f^*(x)e^{\mathrm{i}2\pi sx}\mathrm{d}x = \left[\int_{0}^{\infty} f(x)e^{-\mathrm{i}2\pi sx}\mathrm{d}x\right]^* = F^*(s).$$

Consequently, $G(s) = \tfrac{1}{2}F(s) + \tfrac{1}{2}F^*(s) = F_R(s)$ and $H(s) = \tfrac{1}{2}F(s) - \tfrac{1}{2}F^*(s) = \mathrm{i}F_I(s)$. Now, since $f(x) = 0$ at $x \le 0$, we have $g(x) = \text{sign}(x)h(x)$ and $h(x) = \text{sign}(x)g(x)$. The Fourier transform of $\text{sign}(x)$ is $-\mathrm{i}/(\pi s)$. Therefore, $G(s) = -\mathrm{i}/(\pi s) * H(s)$ or, equivalently, $F_R(s) = (1/\pi s) * F_I(s)$. Similarly, $H(s) = -\mathrm{i}/(\pi s) * G(s)$ or, equivalently, $F_I(s) = -(1/\pi s) * F_R(s)$.