Problem 2) a) The disk-shaped slices at elevation z have radius $\sqrt{R^2 - z^2}$ and thickness $\mathrm{d}z$. The volume of each disk is, therefore, $\pi(R^2 - z^2)\mathrm{d}z$, which, upon integration, yields the volume v of the bowl, as follows:

$$v = \int_{z=-R}^{h-R} \pi(R^2 - z^2)\mathrm{d}z = \pi\{R^2(h - R + R) - ⅓[(h - R)^3 + R^3]\}$$
$$= \pi[\cancel{R^2 h} - ⅓(h^3 - 3Rh^2 + 3\cancel{R^2 h})] = \pi(R - ⅓h)h^2. \quad (1)$$

b) The ring-shaped slices of the bowl have radius $R\sin\theta$ and width $R\mathrm{d}\theta$. Their surface areas are thus given by $2\pi R^2 \sin\theta\, \mathrm{d}\theta$, which, upon integration from $\theta = \theta_0 = \text{arcccos}[(h - R)/R]$ to $\theta = \pi$, yield the total surface area s of the bowl as

$$s = \int_{\theta=\theta_0}^{\pi} 2\pi R^2 \sin\theta\, \mathrm{d}\theta = -2\pi R^2 \cos\theta\big|_{\theta=\theta_0}^{\pi} = -2\pi R^2[-1 - (h - R)/R] = 2\pi Rh. \quad (2)$$

c) The method of Lagrange multipliers requires that the composite function $s(R, h) + \lambda v(R, h)$ be flat at the optimal point (R, h), which is as yet dependent on the Lagrange multiplier λ. Setting the partial derivatives (with respect to R and h) of the composite function to zero, we find

$$\frac{\partial}{\partial R}[2\pi Rh + \lambda\pi(R - ⅓h)h^2] = 0 \quad \rightarrow \quad \pi(2h + \lambda h^2) = 0 \quad \rightarrow \quad h_1 = 0, h_2 = -2/\lambda. \quad (3)$$

$$\frac{\partial}{\partial h}[2\pi Rh + \lambda\pi(R - ⅓h)h^2] = 0 \rightarrow \pi(2R + 2\lambda Rh - \lambda h^2) = 0 \rightarrow R_1 = 0, R_2 = -2/\lambda. \quad (4)$$

Of the two sets of solutions thus obtained, the first (h_1, R_1) is unacceptable, since it results in $v = 0$; see Eq.(1). As for the second solution (h_2, R_2), it yields

$$v = \pi(R - ⅓h)h^2 = \pi\left(-\frac{2}{\lambda} + \frac{2}{3\lambda}\right)\left(\frac{4}{\lambda^2}\right) = -\frac{16\pi}{3\lambda^3} \quad \rightarrow \quad \lambda_0 = -(16\pi/3v)^{⅓}. \quad (5)$$

Upon substituting for λ in the second set of solutions obtained in Eqs.(3) and (4), one arrives at

$$R_2 = h_2 = -2/\lambda_0 = (3v/2\pi)^{⅓}. \quad (6)$$

The above solution represents an exact hemisphere, with $h = R$, $v = 2\pi R^3/3$, and $s = 2\pi R^2$.