Problem 5) a) Plotting the function involves plotting $\ln(x)$ and $[1+\ln(x)]^2$, then inverting the latter function, i.e., plotting $[1+\ln(x)]^{-2}$, and, finally, multiplying $\ln(x)$ and $[1+\ln(x)]^{-2}$, as shown in the figures below.

The limiting value of $f(x)$ as $x \to 0$ may be obtained as follows:

$$f(e^{-n}) = -n/(1-n)^2 \quad \rightarrow \quad lim_{x\to 0}\, f(x) = -\,lim_{n\to\infty}[n/(1-n)^2] = 0.$$

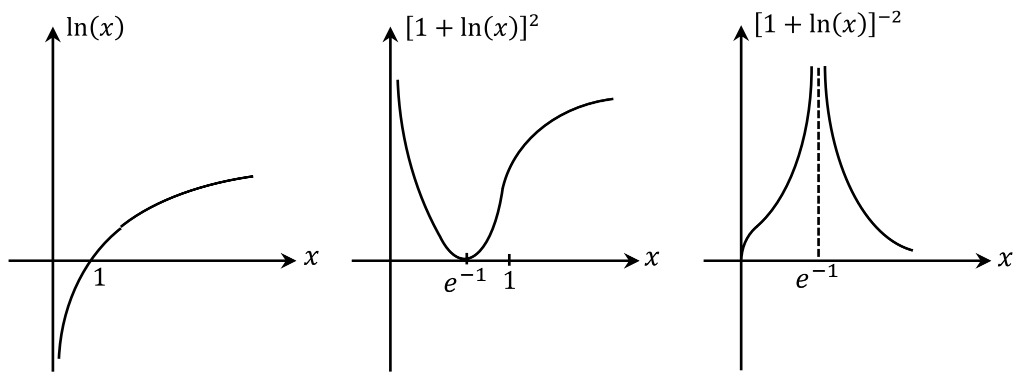
The first derivative of $f(x)$ with respect to x is found to be

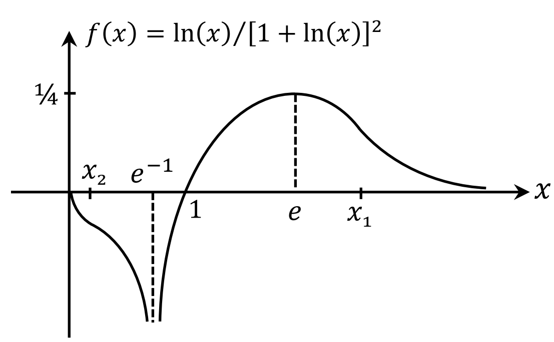
$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\{\ln(x)\,[1+\ln(x)]^{-2}\} = x^{-1}[1+\ln(x)]^{-2} - 2x^{-1}\ln(x)\,[1+\ln(x)]^{-3}$$
$$= x^{-1}[1+\ln(x)]^{-3}[1+\ln(x) - 2\ln(x)] = \frac{1-\ln(x)}{x[1+\ln(x)]^3}.$$

Setting the above derivative to zero, we find $1-\ln(x) = 0$, which yields $x = e$ for the location of the maximum. The peak value of $f(x)$ is then $\ln(e)/[1+\ln(e)]^2 = ¼$. The inflection points are found by setting the second derivative of $f(x)$ to zero; that is,

$$f''(x) = -x^{-2}[1+\ln(x)]^{-3}[1-\ln(x)] - 3x^{-2}[1+\ln(x)]^{-4}[1-\ln(x)] - x^{-2}[1+\ln(x)]^{-3}$$
$$= \frac{\ln^2(x) + 2\ln(x) - 5}{x^2[1+\ln(x)]^4} = 0 \quad \rightarrow \quad \ln^2(x) + 2\ln(x) = 5 \quad \rightarrow \quad x_1 \cong 4.26,\ \ x_2 \cong 0.032.$$

The curvature of $f(x)$ switches sign at the above inflection points. At its peak, the function has a negative curvature equal to $f''(e) = -1/(8e^2)$.





b) The change of variable $y = 1 + \ln(x)$ leads to $\mathrm{d}y/\mathrm{d}x = 1/x$ or, equivalently, $\mathrm{d}x = e^{y-1}\mathrm{d}y$. Consequently,

$$\int_{x=1}^{e} f(x)\mathrm{d}x = \int_{y=1+\ln(1)}^{1+\ln(e)} \frac{(y-1)e^{y-1}}{y^2}\mathrm{d}y = e^{-1}\int_{y=1}^{2} \frac{(y-1)e^{y}}{y^2}\mathrm{d}y$$

$$= e^{-1}\left[-\frac{(y-1)e^{y}}{y}\Big|_{y=1}^{2} + \int_{y=1}^{2} \frac{e^{y}+(y-1)e^{y}}{y}\mathrm{d}y\right] = e^{-1}\left(-½e^2 + \int_1^2 e^y \mathrm{d}y\right)$$

$$= e^{-1}(-½e^2 + e^2 - e) = ½e - 1.$$
