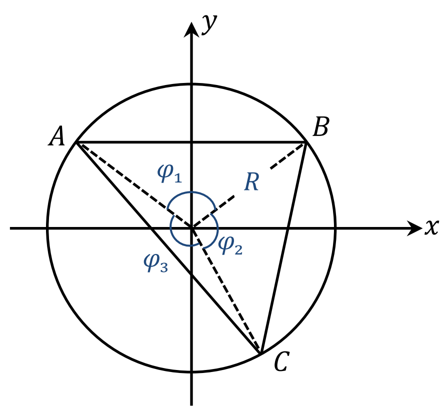
Problem 4) Let the base AB of the triangle be parallel to the x-axis and located at a distance y above the center of the circle. The length of the base will then be $\overline{AB} = 2\sqrt{R^2 - y^2}$. To maximize the area of the triangle, the height h must be maximized, which means that the vertex C should be at $(x, y) = (0, -R)$; this yields $h = R + y$. The area of the triangle is now $½h\overline{AB} = (R + y)\sqrt{R^2 - y^2}$. To find the optimal value of y, differentiate this area with respect to y and set it to zero. You will have

$$\frac{\mathrm{d}}{\mathrm{d}y}(½h\overline{AB}) = \sqrt{R^2 - y^2} - \frac{(R+y)y}{\sqrt{R^2-y^2}} = 0 \quad \rightarrow \quad R^2 - y^2 = Ry + y^2 \quad \rightarrow \quad 2y^2 + Ry - R^2 = 0$$

$$\rightarrow \quad y = \frac{-R \pm \sqrt{R^2+8R^2}}{4} = \frac{-R \pm 3R}{4} \quad \rightarrow \quad y_1 = -R, \quad y_2 = R/2.$$

The 1st solution, $y_1 = -R$, yields a minimum area triangle at $½h\overline{AB} = 0$. The 2nd solution, $y_2 = R/2$, yields the maximum area triangle at $½h\overline{AB} = (R + ½R)\sqrt{R^2 - ¼R^2} = 3\sqrt{3}R^2/4$. At approximately $1.3R^2$, this largest triangle area is still well below the area πR^2 of the circle.

Digression. An alternative solution to this problem is based on constrained optimization via the method of Lagrange multipliers. Let the angles subtended by $\overline{AB}$, $\overline{BC}$, $\overline{AC}$ at the center O of the circle be $\varphi_1, \varphi_2, \varphi_3$, respectively. The area of the triangle OAB is $R^2 \sin(\varphi_1/2)\cos(\varphi_1/2) = ½R^2 \sin(\varphi_1)$. Similarly, the areas of the triangles OBC and OAC will be $½R^2 \sin(\varphi_2)$ and $½R^2 \sin(\varphi_3)$.



Maximizing the area of the triangle ABC thus requires maximizing $\sin(\varphi_1) + \sin(\varphi_2) + \sin(\varphi_3)$ subject to the constraint $\varphi_1 + \varphi_2 + \varphi_3 = 2\pi$. Introducing the Lagrange multiplier λ, we now write

$$\frac{\partial}{\partial\varphi_1}[\sin(\varphi_1) + \sin(\varphi_2) + \sin(\varphi_3) + \lambda(\varphi_1 + \varphi_2 + \varphi_3)] = \cos(\varphi_1) + \lambda = 0,$$

$$\frac{\partial}{\partial\varphi_2}[\sin(\varphi_1) + \sin(\varphi_2) + \sin(\varphi_3) + \lambda(\varphi_1 + \varphi_2 + \varphi_3)] = \cos(\varphi_2) + \lambda = 0,$$

$$\frac{\partial}{\partial\varphi_3}[\sin(\varphi_1) + \sin(\varphi_2) + \sin(\varphi_3) + \lambda(\varphi_1 + \varphi_2 + \varphi_3)] = \cos(\varphi_3) + \lambda = 0.$$

If λ turns out to be negative, all three angles $\varphi_1, \varphi_2, \varphi_3$ can be within the interval $(0, \pi/2)$ and equal to each other, which is impossible considering that their sum must equal 2π. Alternatively, if both φ_1 and φ_2 fall within $(0, \pi/2)$ while φ_3 lands in $(3\pi/2, 2\pi)$, we end up with $\varphi_1 = \varphi_2$ and $\varphi_3 = 2\pi - \varphi_1$, which again makes it impossible for the three angles to add up to 2π. The remaining options for $\varphi_1, \varphi_2, \varphi_3$ can be similarly rejected. Consequently, λ must be positive. In this case, we can have $\varphi_1 = \varphi_2 = \varphi_3 \in (\pi/2\,, \pi)$, which is acceptable, yielding an area for the ABC triangle equal to $½R^2[\sin(\varphi_1) + \sin(\varphi_2) + \sin(\varphi_3)] = (3R^2/2)\sin(2\pi/3)$; this is the same as $3\sqrt{3}R^2/4$ obtained earlier. All other options are unacceptable; for instance, if $\varphi_1 = \varphi_2 \in (\pi/2\,, \pi)$ and $\varphi_3 \in (\pi, 3\pi/2)$, then $\varphi_3 = 2\pi - \varphi_1$, which violates the constraint $\varphi_1 + \varphi_2 + \varphi_3 = 2\pi$.

Constrained optimization via the Lagrange multipliers method can be straightforwardly extended to the case of n-sided polygons enclosed within a circle. It is readily seen that the largest area polygon must be a regular polygon whose sides subtend equal angles $\varphi_1, \varphi_2, \cdots, \varphi_n$ at the center of the circle.
