Opti 403A/503A **Midterm Exam** (3/15/2023) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

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5 pts **Problem 1**) Use the method of proof by induction to show that $\sum_{n=1}^{N} n \cdot (n!) = (N+1)! - 1$.

Problem 2) Consider the geometric series $\sum_{n=0}^{\infty} x^n = 1/(1-x)$, which is valid for $|x| < 1$.

3 pts a) Find an infinite series expansion for $1/(1-x)^2$, valid for $|x| < 1$, by differentiating the geometric series with respect to x.

3 pts b) Obtain the same result as in (a) by directly evaluating $(\sum_{n=0}^{\infty} x^n)^2$.

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5 pts **Problem 3**) Show that $\sin(x) = x \prod_{n=1}^{\infty} \cos(x/2^n)$.

Hint: The identity $\sin(2y) = 2\sin(y)\cos(y)$ will be useful. Also, note that $\sin(y) \to y$ when $y \to 0$.

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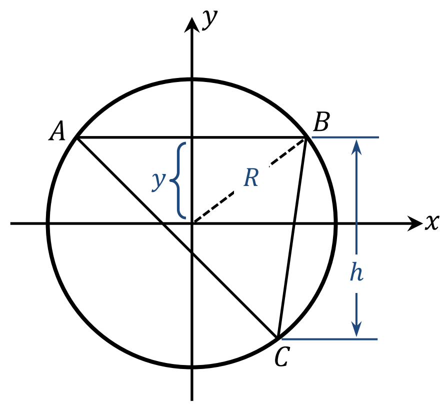
6 pts **Problem 4**) Among all the triangles that can be enclosed within a circle of radius R, find one that has the largest area. Determine the area of this largest triangle as a function of R.

Hint: The base AB of the triangle can be chosen to be parallel to the x-axis. (This is because the symmetry of the circle allows one to rotate the coordinate system around the center until the x-axis becomes parallel to AB.) Fixing the elevation y of the base above the x-axis, choose the location of the vertex C on the perimeter of the circle in such a way as to obtain the maximum height h for the ABC triangle. You must now find the elevation y of the base AB that maximizes the area $½h\overline{AB}$ of the ABC triangle.



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4 pts **Problem 5**) a) Plot a rough sketch of the function $f(x) = \ln(x)/[1+\ln(x)]^2$ over the positive real axis, $x > 0$. Identify the locations of the maxima, minima, and inflection points of the function.

4 pts b) Show that $\int_{x=1}^{e} f(x)\mathrm{d}x = ½e - 1$.

Hint: To evaluate the integral, consider the change of variable $y = 1 + \ln(x)$. The resulting integral may then be computed using the method of integration by parts.

Problem 6) Use Euler's infinite-product expansion of $\sin x$, namely, $\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$, to prove the following identities:

3 pts a) $\cot(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{x^2 - n^2}\right) = \frac{1}{\pi x} + \frac{x}{\pi} \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left[\frac{1}{n(x-n)}\right]$.

2 pts b) $\coth(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{x^2 + n^2}\right)$.

Hint: Convert the infinite product to an infinite sum by taking the natural logarithm of $\sin(x)$. Differentiation with respect to x then yields the infinite series expansion of the cotangent function.
