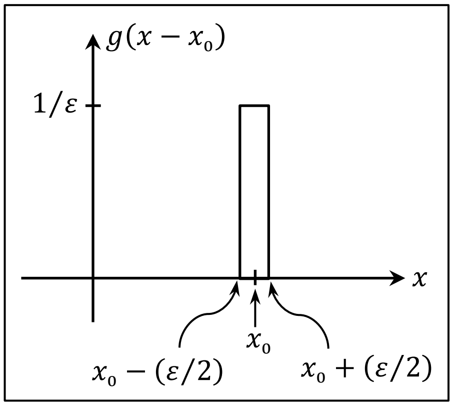
Opti 403A/503A **Final Exam** (5/10/2023) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

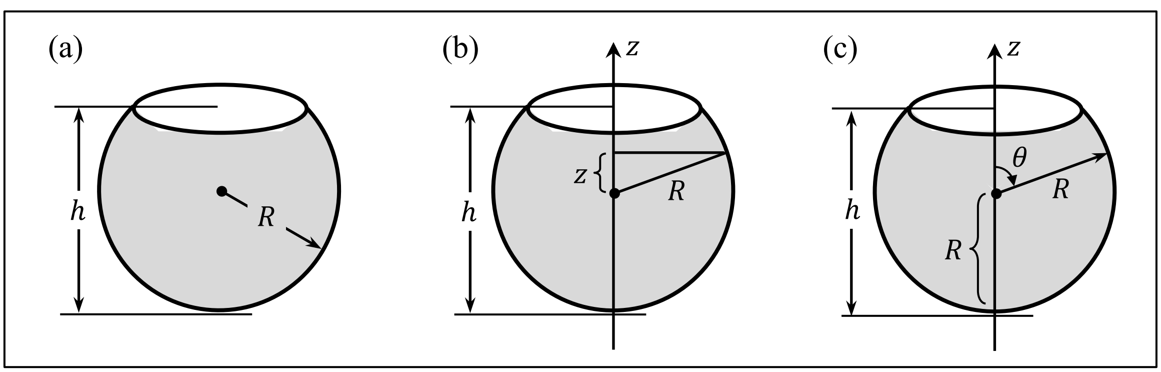
Problem 1) As depicted in the figure, the unit delta-function $\delta(x)$ can be considered to be the limit of a tall, narrow, symmetric rectangular pulse $g(x) = \varepsilon^{-1}\text{rect}(x/\varepsilon)$, whose area equals 1.



6 pts a) Write an expression (consisting of a pair of shifted delta-functions) for the derivative with respect to x of $g(x)$, namely, $g'(x) = \mathrm{d}[\varepsilon^{-1}\text{rect}(x/\varepsilon)]/\mathrm{d}x$.

6 pts b) Show that, for any well-behaved function $f(x)$ that is continuous and differentiable in the vicinity of an arbitrary point x_0, the function $g'(x)$ obtained in part (a) exhibits the characteristic sifting property of $\delta'(x)$; that is, $\int_{-\infty}^{\infty} f(x)g'(x - x_0)\mathrm{d}x = -f'(x_0)$.

Problem 2) A hollow spherical shell of radius R is decapitated at a height h to create a bowl of volume v, as shown in figure (a). In this problem you are asked to use the method of Lagrange multipliers to determine the values of R and h that produce the smallest surface area s for some desired volume v.



4 pts a) Derive an expression for the volume v of the bowl by integrating over disk-shaped slices at elevation z (relative to the center of the sphere) from $z = -R$ to $z = h - R$; see figure (b).

4 pts b) Derive an expression for the surface area s of the bowl by integrating over rings of radius $R\sin\theta$, where θ ranges from $\theta_0 = \arccos[(h - R)/R]$ to π; see figure (c).

continued on the next page…

6 pts c) Use the method of Lagrange multipliers to find the values of R and h for the smallest surface area s that is compatible with a desired volume v of the bowl.

Hint: For part (a), the identity $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$ can be helpful. For part (b), the needed integral is $\int \sin\theta \, d\theta = -\cos\theta$. In part (c), the method starts with forming the compound function $s(R,h) + \lambda v(R,h)$, where the adjustable parameter λ is the Lagrange multiplier.

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12 pts **Problem 3**) A function $f(x)$ has nonzero values only over the positive half of the x-axis; that is, $f(x) = 0$ for $x \le 0$. Multiplying $f(x)$ by the unit step-function $\text{step}(x)$ does not change $f(x)$ in any way; therefore, $\text{step}(x)f(x) = f(x)$. Let the Fourier transform $F(s)$ of $f(x)$ be written in terms of its real and imaginary parts, namely, $F(s) = F_R(s) + \mathrm{i}F_I(s)$. Considering that the Fourier transform of $\text{step}(x)$ is $\tfrac{1}{2}\delta(s) - \mathrm{i}/(2\pi s)$, find the relationship between $F_R(s)$ and $F_I(s)$.

Hint: You will find that $F_R(s)$ is the convolution between $F_I(s)$ and a third function. Similarly, $F_I(s)$ turns out to be the convolution between $F_R(s)$ and a third function.

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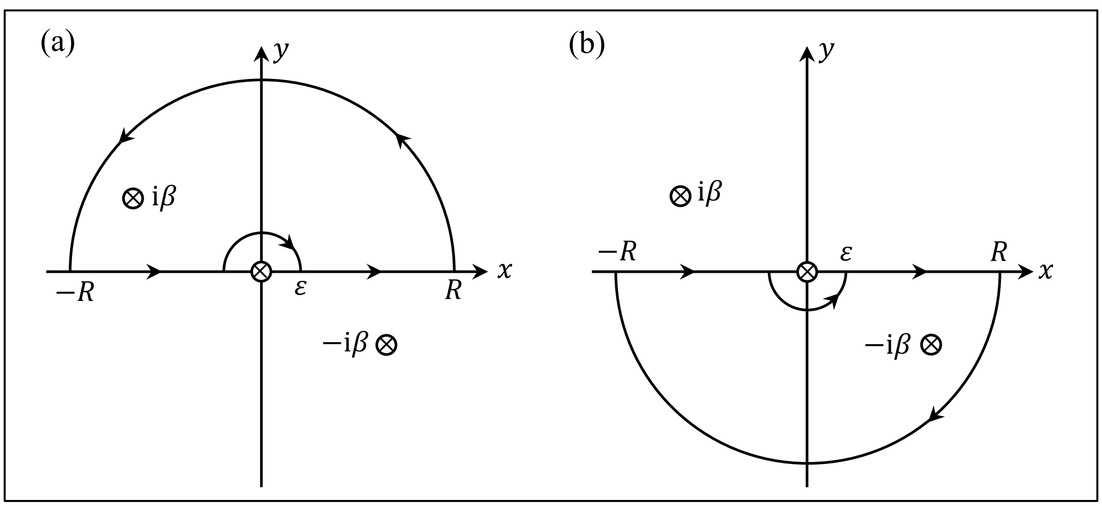
12 pts **Problem 4**) Given the constant parameters $a > 0$, $b > 0$ and $\beta = |\beta|e^{\mathrm{i}\varphi_\beta}$, where $0 \le \varphi_\beta < \pi/2$ and $|\beta| \ne 0$, use complex-plane techniques to show that

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2(x^2+\beta^2)} \mathrm{d}x = \frac{\pi\left[(b-a)\beta + e^{-b\beta} - e^{-a\beta}\right]}{2\beta^3}.$$

You must indicate how the integral is extended to the entire x-axis at first, and also why the large semi-circles do *not* contribute to the corresponding loop integral. (Citation of the relevant theorem or lemma will suffice.) The necessary residues at each pole must be clearly evaluated.



Hint: You may invoke the identity $\cos(\zeta) = \tfrac{1}{2}(e^{\mathrm{i}\zeta} + e^{-\mathrm{i}\zeta})$, then use the contours shown in the figure to evaluate the resulting integrals. Note that $z = 0$ is a second-order pole of the integrand.
