Problem 4) a) The convolution integral must first be converted to a form suitable for complex-plane methods, as follows:

$$\begin{aligned}\mathrm{sinc}(\alpha x) * \mathrm{sinc}(\beta x) &= \int_{-\infty}^{\infty} \mathrm{sinc}(\alpha x')\mathrm{sinc}[\beta(x-x')]\mathrm{d}x' \\ &= \int_{-\infty}^{\infty}\left[\frac{e^{\mathrm{i}\pi\alpha x'} - e^{-\mathrm{i}\pi\alpha x'}}{\mathrm{i}2\pi\alpha x'} \times \frac{e^{\mathrm{i}\pi\beta(x-x')} - e^{-\mathrm{i}\pi\beta(x-x')}}{\mathrm{i}2\pi\beta(x-x')}\right]\mathrm{d}x' \\ &= \frac{1}{4\pi^2\alpha\beta}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}\pi\beta x}[e^{\mathrm{i}\pi(\alpha-\beta)x'} - e^{-\mathrm{i}\pi(\alpha+\beta)x'}] + e^{-\mathrm{i}\pi\beta x}[e^{-\mathrm{i}\pi(\alpha-\beta)x'} - e^{\mathrm{i}\pi(\alpha+\beta)x'}]}{x'(x'-x)}\mathrm{d}x'.\end{aligned} \tag{1}$$

Considering that the two terms in the above integrand are complex-conjugates, we proceed to evaluate only the first half of the integral, then combine the end result with its conjugate. The first half of the integral has two terms, which are found by complex-plane integration, namely,

$$\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}\pi(\alpha-\beta)x'}}{x'(x'-x)}\mathrm{d}x' = \mathrm{i}\pi\left[-\frac{1}{x} + \frac{e^{\mathrm{i}\pi(\alpha-\beta)x}}{x}\right]. \tag{2}$$

$$\int_{-\infty}^{\infty}\frac{e^{-\mathrm{i}\pi(\alpha+\beta)x'}}{x'(x'-x)}\mathrm{d}x' = -\mathrm{i}\pi\left[-\frac{1}{x} + \frac{e^{-\mathrm{i}\pi(\alpha+\beta)x}}{x}\right]. \tag{3}$$

The integral in Eq.(2) is evaluated in the upper-half plane, and that in Eq.(3) in the lower-half plane. In both cases, the integrals equal the sum of two half-residues at $x' = 0$ and $x' = x$. The first half of the expression on the right-hand side of Eq.(1) is now found to be

$$\begin{aligned}\frac{e^{\mathrm{i}\pi\beta x}}{4\pi^2\alpha\beta}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}\pi(\alpha-\beta)x'} - e^{-\mathrm{i}\pi(\alpha+\beta)x'}}{x'(x'-x)}\mathrm{d}x' &= \frac{\mathrm{i}\pi e^{\mathrm{i}\pi\beta x}}{4\pi^2\alpha\beta}\left[-\frac{2}{x} + \frac{e^{\mathrm{i}\pi(\alpha-\beta)x}}{x} + \frac{e^{-\mathrm{i}\pi(\alpha+\beta)x}}{x}\right] \\ &= \frac{\mathrm{i}(-2e^{\mathrm{i}\pi\beta x} + e^{\mathrm{i}\pi\alpha x} + e^{-\mathrm{i}\pi\alpha x})}{4\pi\alpha\beta x} = \frac{\mathrm{i}[\cos(\pi\alpha x) - e^{\mathrm{i}\pi\beta x}]}{2\pi\alpha\beta x}.\end{aligned} \tag{4}$$

Adding the above expression to its complex-conjugate yields the final result of the convolution integral, as follows:

$$\mathrm{sinc}(\alpha x) * \mathrm{sinc}(\beta x) = \frac{\mathrm{i}[\cos(\pi\alpha x) - e^{\mathrm{i}\pi\beta x}]}{2\pi\alpha\beta x} - \frac{\mathrm{i}[\cos(\pi\alpha x) - e^{-\mathrm{i}\pi\beta x}]}{2\pi\alpha\beta x} = \frac{\sin(\pi\beta x)}{\pi\alpha\beta x} = \alpha^{-1}\mathrm{sinc}(\beta x). \tag{5}$$

b) Using the scaling theorem of Fourier transformation, we have $\mathcal{F}\{\mathrm{sinc}(\alpha x)\} = |\alpha|^{-1}\mathrm{rect}(s/\alpha)$ and $\mathcal{F}\{\mathrm{sinc}(\beta x)\} = |\beta|^{-1}\mathrm{rect}(s/\beta)$. The convolution theorem now yields

$$\mathcal{F}\{\mathrm{sinc}(\alpha x) * \mathrm{sinc}(\beta x)\} = \alpha^{-1}\mathrm{rect}(s/\alpha) \times \beta^{-1}\mathrm{rect}(s/\beta) = (\alpha\beta)^{-1}\mathrm{rect}(s/\beta). \tag{6}$$

Here, we have used the fact that the product of two $rect$ functions equals the narrower of the two, which, in the present case is $\mathrm{rect}(s/\beta)$. Upon inverse Fourier transformation, we arrive at

$$\mathcal{F}^{-1}\{(\alpha\beta)^{-1}\mathrm{rect}(s/\beta)\} = \alpha^{-1}\mathcal{F}^{-1}\{\beta^{-1}\mathrm{rect}(s/\beta)\} = \alpha^{-1}\mathrm{sinc}(\beta x). \tag{7}$$