Problem 3) a) $\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \frac{\mathrm{d}\eta(x)}{\mathrm{d}x} \times \frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta} = -x^{-2}\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}$.

$$x^2 f'(x) + f(x) = 0 \quad \rightarrow \quad -\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta} + f(\eta) = 0 \quad \rightarrow \quad f(\eta) = Ae^{\eta} \quad \rightarrow \quad f(x) = Ae^{1/x}.$$

b) When $x \rightarrow \pm\infty$, we will have $f(x) \rightarrow Ae^0 = A$. When $x \rightarrow 0^-$, we find $f(x) \rightarrow Ae^{-\infty} = 0$. and when $x \rightarrow 0^+$, we will have $f(x) \rightarrow Ae^{+\infty} = \infty$. Note also that, in the limit when $x \rightarrow 0^-$, all the derivatives of $f(x)$ approach zero. For example, $lim_{x\rightarrow 0^-} f'(x) = lim_{x\rightarrow 0^-}\left(-x^{-2}e^{1/x}\right) = 0$. Thus, in the vicinity of $x = 0^-$, the function $f(x)$ is extremely smooth, since all its derivatives approach zero when x approaches the origin from the left-hand side. A sketch of $e^{1/x}$ is shown below. The inflection point at $x = -½$ is where the second derivative of the function vanishes, that is, where $\mathrm{d}^2(e^{1/x})/\mathrm{d}x^2 = (2x^{-3} + x^{-4})e^{1/x} = 0$. Note that the graph is not properly scaled.