Problem 2) The binomial expansion of $\cos^{2n}(\pi x) = (e^{\mathrm{i}\pi x} + e^{-\mathrm{i}\pi x})^{2n}/2^{2n}$ yields

$$\cos^{2n}(\pi x) = \left(\frac{e^{\mathrm{i}\pi x}+e^{-\mathrm{i}\pi x}}{2}\right)^{2n} = \frac{1}{2^{2n}}\sum_{m=0}^{2n}\binom{2n}{m}\left(e^{\mathrm{i}\pi x}\right)^{2n-m}\left(e^{-\mathrm{i}\pi x}\right)^{m}$$

$$= \frac{1}{4^n}\sum_{m=0}^{2n}\binom{2n}{m}e^{\mathrm{i}2\pi(n-m)x} = \frac{1}{4^n}\sum_{k=-n}^{n}\binom{2n}{n-k}e^{\mathrm{i}2\pi kx}. \quad (1)$$

$n - m = k$

Recalling that the Fourier transform of $e^{\mathrm{i}2\pi s_0 x}$ is $\delta(s - s_0)$, we find the Fourier transform of the above function to be

$$\mathcal{F}\{\cos^{2n}(\pi x)\} = \frac{1}{4^n}\sum_{k=-n}^{n}\binom{2n}{n-k}\int_{-\infty}^{\infty}e^{\mathrm{i}2\pi kx}e^{-\mathrm{i}2\pi sx}\mathrm{d}x = \frac{1}{4^n}\sum_{k=-n}^{2n}\binom{2n}{n-k}\delta(s-k). \quad (2)$$

Upon normalization by A_n, the magnitude of the k^{th} delta-function in the Fourier domain will be

$$\frac{(2n)!!}{4^n(2n-1)!!}\binom{2n}{n-k} = \frac{2^n(n!)}{4^n(2n-1)!!}\times\frac{(2n)!}{(n-k)!(n+k)!} = \frac{(n!)(2n)!!}{2^n(n-k)!(n+k)!} = \frac{(n!)^2}{(n-k)!(n+k)!}. \quad (3)$$

It is seen that the central δ-function located at $s = 0$ (corresponding to $k = 0$) has unit magnitude. The adjacent δ-functions at $s = \pm 1$ have a slightly reduced magnitude $n/(n + 1)$. At $s = \pm 2$, the magnitude of both δ-functions is $n(n-1)/[(n+1)(n+2)]$, and so on. The decline continues until $s = \pm n$, where the terminal δ-functions have the negligible magnitude

$$\frac{(n!)^2}{(2n)!} = \frac{(n!)^2}{(2n)!!(2n-1)!!} = \frac{n!}{2^n(2n-1)!!} = \frac{(2n)!!}{2^{2n}(2n-1)!!}. \quad (4)$$

As n grows larger, the set of δ-functions in the Fourier domain spreads further out from the center while its central δ-functions become more uniform in magnitude. In the limit of $n \to \infty$, the Fourier transform of the normalized $\cos^{2n}(\pi x)$ approaches $\sum_{k=-\infty}^{\infty}\delta(s-k) = \mathrm{comb}(s)$.

Digression: The normalization factor has been determined via integration by parts, as follows:

$$\int_{-½}^{½}\cos^{2n}(\pi x)\,\mathrm{d}x = \int_{-½}^{½}\cos(\pi x)\cos^{2n-1}(\pi x)\,\mathrm{d}x$$

$$= \pi^{-1}\sin(\pi x)\cos^{2n-1}(\pi x)\Big|_{-½}^{½} + (2n-1)\int_{-½}^{½}\sin^2(\pi x)\cos^{2n-2}(\pi x)\,\mathrm{d}x$$

$$= (2n-1)\int_{-½}^{½}[1-\cos^2(\pi x)]\cos^{2n-2}(\pi x)\,\mathrm{d}x$$

$$= (2n-1)\int_{-½}^{½}\cos^{2(n-1)}(\pi x)\,\mathrm{d}x - (2n-1)\int_{-½}^{½}\cos^{2n}(\pi x)\,\mathrm{d}x$$

$$\to \quad \int_{-½}^{½}\cos^{2n}(\pi x)\,\mathrm{d}x = \left(\frac{2n-1}{2n}\right)\int_{-½}^{½}\cos^{2(n-1)}(\pi x)\,\mathrm{d}x. \quad (5)$$

Repeating the above process, we note that, in each step, n in the exponent of $\cos^{2n}(\pi x)$ is reduced by 1. Eventually we will reach $\cos^0(\pi x) = 1$, whose integral equals 1. Consequently,

$$\int_{-½}^{½}\cos^{2n}(\pi x)\,\mathrm{d}x = \frac{(2n-1)(2n-3)\cdots(3)(1)}{2n(2n-2)\cdots(4)(2)} = \frac{(2n-1)!!}{(2n)!!}. \quad (6)$$