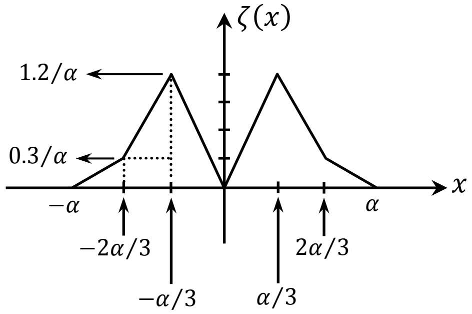
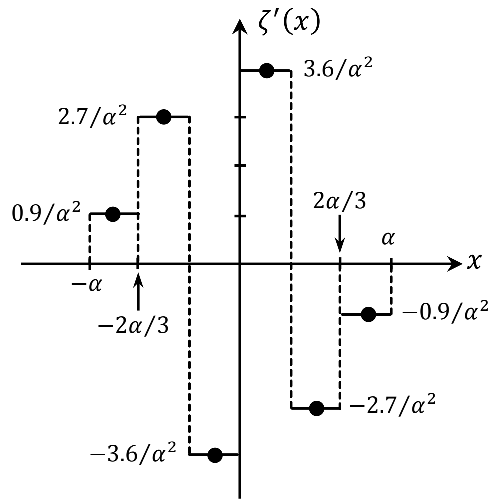
Problem 1) a) The area under $\zeta(x)$ is $2(\alpha/3)[(0.3/\alpha) + (1.2/\alpha)] = 1$. The function is even (i.e., symmetric around the vertical axis) and, in the limit when $\alpha \to 0$, it becomes tall and narrow. These are all the requirements for a δ-function. Consequently, $lim_{\alpha\to 0}\, \zeta(x) = \delta(x)$.

b) A plot of the derivative $\zeta'(x)$ of $\zeta(x)$ is shown below. For an arbitrary function $f(x)$ that is continuous and well-behaved in the vicinity of $x = 0$, the sifting property of $\zeta'(x)$ when α is sufficiently small is verified by integrating $f(x)\zeta'(x)$ over all six segments of $\zeta'(x)$, as follows:





$$\int_{-\infty}^{\infty} f(x)\zeta'(x)\mathrm{d}x = (\alpha/3)\,[(0.9/\alpha^2)f(-5\alpha/6) + (2.7/\alpha^2)f(-\alpha/2) - (3.6/\alpha^2)f(-\alpha/6)$$
$$+(3.6/\alpha^2)f(\alpha/6) - (2.7/\alpha^2)f(\alpha/2) - (0.9/\alpha^2)f(5\alpha/6)]$$
$$= (0.3/\alpha)[f(-5\alpha/6) - f(5\alpha/6)] + (0.9/\alpha)[f(-\alpha/2) - f(\alpha/2)]$$
$$-(1.2/\alpha)[f(-\alpha/6) - f(\alpha/6)]$$
$$= -(0.3/\alpha)(5\alpha/3)f'(0) - (0.9/\alpha)\alpha f'(0) + (1.2/\alpha)(\alpha/3)f'(0)$$
$$= (-0.5 - 0.9 + 0.4)f'(0) = -f'(0).$$