Opti 403A/503A **Final Exam** (5/11/2022) **Time: 2 hours**

Please write your name and ID number on all the pages, then staple them together. Answer all the questions.

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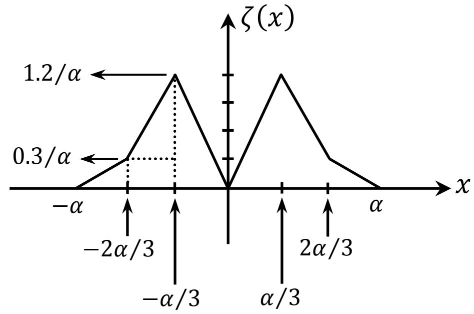
6 pts **Problem 1**) a) Explain why the function $\zeta(x)$ shown in the figure approaches a Dirac delta-function $\delta(x)$ in the limit when $\alpha \to 0$.



8 pts b) Plot the function $\zeta'(x) = \mathrm{d}\zeta(x)/\mathrm{d}x$, then argue that $\zeta'(x)$ approaches the derivative $\delta'(x)$ of $\delta(x)$ in the limit when $\alpha \to 0$.

Hint: You must demonstrate the sifting property of $\delta'(x)$; that is, $\int_{-\infty}^{\infty} f(x)\delta'(x)\mathrm{d}x = -f'(0)$ for an arbitrary function $f(x)$ that is continuous and well-behaved in the vicinity of $x = 0$.

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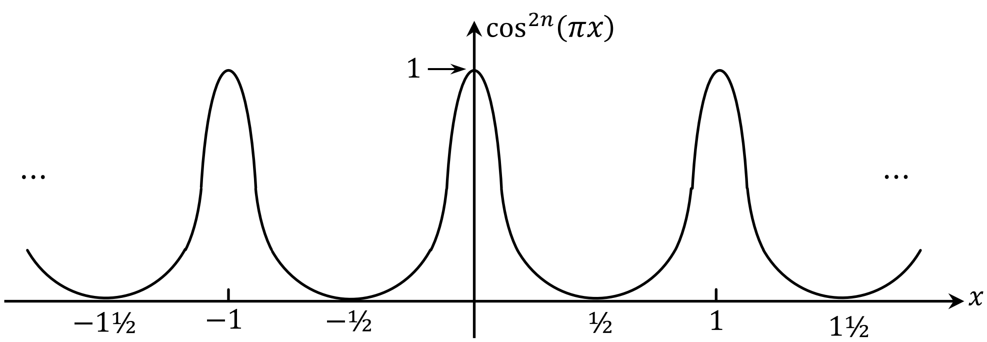
10 pts **Problem 2**) The function $f(x) = \cos^2(\pi x)$ is periodic, with period $p = 1$. When raised to the power of a large positive integer n, the function resembles a periodic array of narrow and symmetric pulses of unit height, as seen below. The area under each pulse can be shown to be

$$A_n = \int_{-½}^{½} \cos^{2n}(\pi x)\,\mathrm{d}x = \frac{(2n-1)!!}{(2n)!!}.$$

Show that the Fourier transform of $A_n^{-1}\cos^{2n}(\pi x)$ approaches a *comb* function when $n \to \infty$; that is, $lim_{n\to\infty}\mathcal{F}\{(2n)!!\cos^{2n}(\pi x)/(2n-1)!!\} = \mathrm{comb}(s)$.



Hint: Use $\cos(\pi x) = (e^{\mathrm{i}\pi x} + e^{-\mathrm{i}\pi x})/2$ and the binomial expansion $(a+b)^n = \sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^m$. Also, recall that $\mathcal{F}\{e^{\mathrm{i}2\pi s_0 x}\} = \int_{-\infty}^{\infty} e^{\mathrm{i}2\pi s_0 x}e^{-\mathrm{i}2\pi s x}\mathrm{d}x = \delta(s - s_0)$.

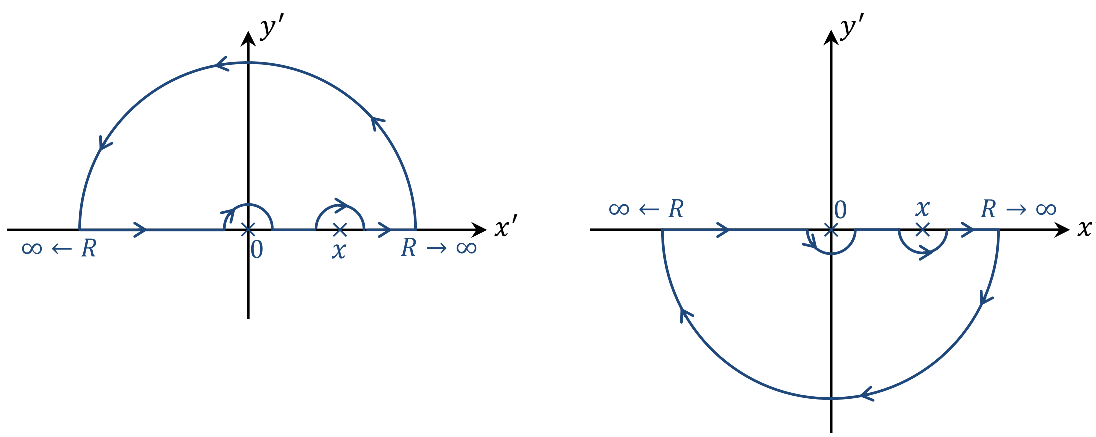
Problem 3) One way to solve the first-order ordinary differential equation $x^2 f'(x) + f(x) = 0$ is to change the independent variable from x to $\eta = x^{-1}$, then use the chain rule of differentiation to transform the above equation into a 1st order ODE with constant coefficients for $f(\eta)$. The resulting ODE is easy to solve and substitution for η in $f(\eta)$ would yield the desired solution $f(x)$.

6 pts a) Use the above method to find the solution of the differential equation.

6 pts b) Plot the function $f(x)$ over the entire x-axis, paying particular attention to the neighborhood of $x = 0$ and also to the behavior of $f(x)$ when $x \to \pm\infty$.

Problem 4) It is well known that the convolution of two *sinc* functions yields another *sinc* function. Specifically, $\text{sinc}(\alpha x) * \text{sinc}(\beta x) = \alpha^{-1}\text{sinc}(\beta x)$, where α and β are real numbers satisfying $\alpha \geq \beta > 0$.

8 pts a) Write $\text{sinc}(\alpha x) * \text{sinc}(\beta x)$ as a standard convolution integral, then evaluate the integral using complex-plane methods based on the Cauchy-Goursat theorem. (Suggested contours of integration are shown below. The end result of this calculation should be $\alpha^{-1}\text{sinc}(\beta x)$.)



6 pts b) Use the convolution theorem and the scaling theorem of the Fourier transform theory to demonstrate in a much simpler way that indeed $\text{sinc}(\alpha x) * \text{sinc}(\beta x) = \alpha^{-1}\text{sinc}(\beta x)$.

Hint: $\text{sinc}(x) = \sin(\pi x)/(\pi x) = (e^{i\pi x} - e^{-i\pi x})/(i2\pi x)$. Also, $\mathcal{F}\{\text{sinc}(x)\} = \text{rect}(s)$.
