Problem 3) a) $r(\varphi) = Re^{-\alpha\varphi} \quad \rightarrow \quad \mathrm{d}r = -\alpha Re^{-\alpha\varphi}\mathrm{d}\varphi = -\alpha r(\varphi)\mathrm{d}\varphi$

$$\rightarrow \quad \mathrm{d}\ell = \sqrt{(r\mathrm{d}\varphi)^2 + (\mathrm{d}r)^2} = \sqrt{1+\alpha^2}\, r(\varphi)\mathrm{d}\varphi. \qquad (1)$$

b) The total length ℓ of the spiral is given by

$$\ell = \int_{\varphi=0}^{\infty} \mathrm{d}\ell = \int_0^{\infty} \sqrt{1+\alpha^2}\, r(\varphi)\mathrm{d}\varphi = \sqrt{1+\alpha^2}R \int_0^{\infty} e^{-\alpha\varphi}\mathrm{d}\varphi$$

$$= -\sqrt{1+\alpha^2}R\alpha^{-1}e^{-\alpha\varphi}\Big|_{\varphi=0}^{\infty} = \sqrt{1+\alpha^{-2}}R. \qquad (2)$$

c) In accordance with Eq.(2), for large values of α, the length ℓ of the spiral is only slightly greater than R; that is, $\ell \cong R + ½(R/\alpha^2)$, whereas for small values of α, the length ℓ very nearly equals R/α.