Problem 2) The first-order partial derivatives of $f(x, y)$ are

$$\partial_x f = \frac{\partial f}{\partial x} = y(1 - 4x^2)e^{-2(x^2+y^2)}, \quad (1)$$

$$\partial_y f = \frac{\partial f}{\partial y} = x(1 - 4y^2)e^{-2(x^2+y^2)}. \quad (2)$$

Setting both $\partial_x f$ and $\partial_y f$ equal to zero, we find five different solutions that represent the locations of the minima, maxima, and saddle-points of $f(x, y)$, namely,

$$\partial_x f = \partial_y f = 0 \quad \rightarrow \quad (x_0, y_0) = (0, 0); \quad (x_1, y_1) = (½, ½); \quad (x_2, y_2) = (½, -½);$$
$$(x_3, y_3) = (-½, ½); \quad (x_4, y_4) = (-½, -½). \quad (3)$$

To determine which one of the five points in Eq.(3) corresponds to a minimum, maximum, or saddle point, we need to compute the second-order partial derivatives of $f(x, y)$, as follows:

$$\partial_{xx} f = \frac{\partial^2 f}{\partial x^2} = -[8xy + 4xy(1 - 4x^2)]e^{-2(x^2+y^2)} = 4xy(4x^2 - 3)e^{-2(x^2+y^2)}, \quad (4)$$

$$\partial_{yy} f = \frac{\partial^2 f}{\partial y^2} = -[8xy + 4xy(1 - 4y^2)]e^{-2(x^2+y^2)} = 4xy(4y^2 - 3)e^{-2(x^2+y^2)}, \quad (5)$$

$$\partial_{xy} f = \partial_{yx} f = \frac{\partial^2 f}{\partial x \partial y} = (4x^2 - 1)(4y^2 - 1)e^{-2(x^2+y^2)}. \quad (6)$$

These second-order partial derivatives, when evaluated at the five points listed in Eq.(3), yield

At $(x_0, y_0) = (0, 0)$: $\quad \partial_{xx} f|_{(x_0,y_0)} = 0, \quad \partial_{yy} f\big|_{(x_0,y_0)} = 0, \quad \partial_{xy} f\big|_{(x_0,y_0)} = 1. \quad (7)$

At $(x_1, y_1) = (½, ½)$: $\quad \partial_{xx} f|_{(x_1,y_1)} = -2e^{-1}, \quad \partial_{yy} f\big|_{(x_1,y_1)} = -2e^{-1}, \quad \partial_{xy} f\big|_{(x_1,y_1)} = 0. \quad (8)$

At $(x_2, y_2) = (½, -½)$: $\quad \partial_{xx} f|_{(x_2,y_2)} = 2e^{-1}, \quad \partial_{yy} f\big|_{(x_2,y_2)} = 2e^{-1}, \quad \partial_{xy} f\big|_{(x_2,y_2)} = 0. \quad (9)$

At $(x_3, y_3) = (-½, ½)$: $\quad \partial_{xx} f|_{(x_3,y_3)} = 2e^{-1}, \quad \partial_{yy} f\big|_{(x_3,y_3)} = 2e^{-1}, \quad \partial_{xy} f\big|_{(x_3,y_3)} = 0. \quad (10)$

At $(x_4, y_4) = (-½, -½)$: $\quad \partial_{xx} f|_{(x_4,y_4)} = -2e^{-1}, \quad \partial_{yy} f\big|_{(x_4,y_4)} = -2e^{-1}, \quad \partial_{xy} f\big|_{(x_4,y_4)} = 0. \quad (11)$

There will be a peak (maximum) or a valley (minimum) wherever $(\partial_{xx} f)(\partial_{yy} f) > (\partial_{xy} f)^2$. This means that (x_0, y_0) is neither the location of a peak nor a valley, but that the remaining four points are. At (x_1, y_1) and (x_4, y_4), we have $\partial_{xx} f < 0$ (and also $\partial_{yy} f < 0$); consequently, both of these points correspond to peaks (or maxima) of $f(x, y)$. In contrast, (x_2, y_2) and (x_3, y_3) are locations of valleys (or minima) of $f(x, y)$, since, at both points, $\partial_{xx} f > 0$ (and also $\partial_{yy} f > 0$).

As for $(x_0, y_0) = (0, 0)$, given that $(\partial_{xx} f)(\partial_{yy} f) < (\partial_{xy} f)^2$ at this point, it must be the location of a saddle point. Now, in the vicinity of (x_0, y_0), we can write

$$\Delta f = f(x, y) - f(x_0, y_0) = (\partial_x f)(x - x_0) + (\partial_y f)(y - y_0) + ½(\partial_{xx} f)(x - x_0)^2 +$$
$$+½(\partial_{yy} f)(y - y_0)^2 + (\partial_{xy} f)(x - x_0)(y - y_0) + \cdots \cong xy. \quad (12)$$

It is seen that, in the 1st and 3rd quadrants of the xy-plane, where $xy > 0$, we have $\Delta f > 0$, which indicates that $f(x, y)$ rises above $f(x_0, y_0) = f(0,0) = 0$. In contrast, in the 2nd and 4th quadrants, where $xy < 0$, we have $\Delta f < 0$, indicating that $f(x, y)$ drops below $f(x_0, y_0)$.