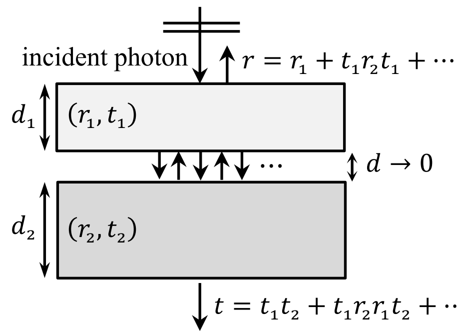
Opti 403A/503A **Midterm Exam** (3/16/2022) **Time: 75 minutes**

Please write your name and ID number on all the pages, then staple them together.
Answer all the questions.

Problem 1) According to the laws of quantum optics, a single photon has a probability amplitude r of being reflected from a slab of material, and also another probability amplitude t of being transmitted through the slab. (Typically, r and t are complex numbers.) The figure shows two back-to-back slabs, one with probability amplitudes r_1 and t_1, the other with probability amplitudes r_2 and t_2. When the slabs come into contact with each other to form a bilayer, the gap-width d between them approaches zero, so that no phase-shifts need be associated with propagations that take place inside the gap. In principle, each path that the photon could possibly take through the pair of slabs is permissible and, in the end, all probability amplitudes corresponding to different paths leading to a specific event must be added together. For instance, reflection from the top of the bilayer could occur with probability amplitude r_1, corresponding to a direct reflection, as well as $t_1r_2t_1$, corresponding to transmission through the first slab, reflection at the surface of the second slab, and a subsequent transmission (from below) through the first slab again. Many such possibilities exist, and their probability amplitudes must all be added together to produce the overall amplitude for reflection from the top of the bilayer slab.



4 pts a) Write the infinite sum associated with reflection from the top of the bilayer, then find a closed form for the resulting series, which will be a formula for the bilayer's upper facet reflection coefficient r in terms of r_1, t_1, and r_2.

4 pts b) Write the infinite sum associated with transmission through the bilayer, then find a closed form for the resulting series, which will be a formula for the bilayer's transmission coefficient t in terms of r_1, t_1, r_2, and t_2.

2 pts c) Based on the result of part (b), explain why the bilayer is expected to have the same transmission coefficient t irrespective of whether the photon is incident at the top or at the bottom facet of the bilayer.

Hint: In general, each slab could have different (r, t) values for incidence from its top and bottom sides. However, it is being assumed in this problem that (r_1, t_1) are the same for incidence on the top and bottom facets of the first slab; similarly, (r_2, t_2) are assumed to be the same for incidence on the top and bottom facets of the second slab. The methods used in this problem can be applied to normal as well as oblique incidence, provided that all (r, t) values in any given situation correspond to the same direction of incidence and to certain states of polarization of the photon.

---

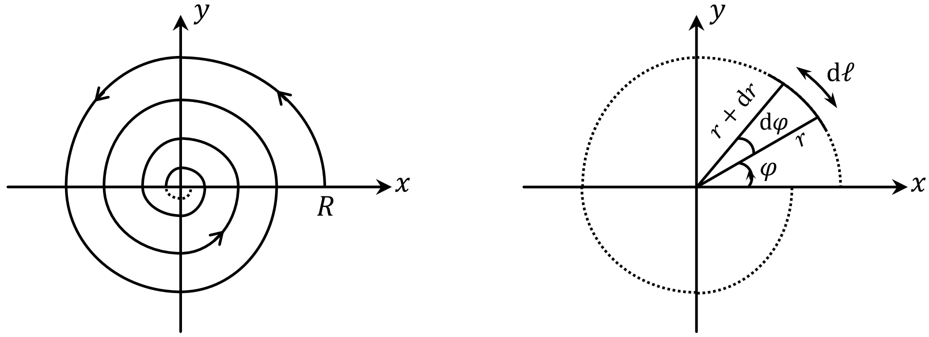
4 pts **Problem 2**) a) Find all the minima, maxima, and saddle points of the following function of two variables:

$$f(x, y) = xye^{-2(x^2+y^2)}.$$

4 pts b) In the vicinity of the saddle point (x_0, y_0), identify the region of the xy-plane where $f(x, y)$ drops below $f(x_0, y_0)$, and also the region where $f(x, y)$ rises above $f(x_0, y_0)$.

··· continued on the backside

Problem 3) A spiral is the path of a point-particle moving in the xy-plane, whose distance r from the origin continuously shrinks with an increasing polar angle φ, as shown in the figure. For the so-called logarithmic spiral, $r(\varphi) = Re^{-\alpha\varphi}$, where R and α are positive constants.



4 pts a) Use the theorem of Pythagoras (i.e., $a^2 + b^2 = c^2$ for right-angle triangles) to express the differential length $\mathrm{d}\ell$ of the logarithmic spiral as a function of R, α, φ, and $\mathrm{d}\varphi$.

3 pts b) Determine the total length ℓ of the logarithmic spiral by integrating $\mathrm{d}\ell$ over φ from 0 to ∞.

2 pts c) Find good approximations for the length ℓ of the spiral when $\alpha \gg 1$, and also when $\alpha \ll 1$.

---

8 pts **Problem 4**) Consider the arbitrary triangle ABC, whose sides have lengths a, b, c, and whose angles are θ, φ, and $(\pi - \theta - \varphi)$, as shown in the figure. The altitude CD dropped from C onto AB has length h_1, and the normal DE dropped from D onto BC has length h_2, and so on. Find a closed form expression for the infinite sum $h_1 + h_2 + h_3 + \cdots$ in terms of b and the angles θ and φ.

Hint: The angle identified in the figure as ψ is the complement of φ; that is, $\varphi + \psi = \frac{1}{2}\pi$.