Problem 5)

a) $f_1(z)=\dfrac{1}{z-z_o}=\dfrac{1}{(x-x_o)+\mathrm{i}(y-y_o)}=\dfrac{(x-x_o)-\mathrm{i}(y-y_o)}{(x-x_o)^2+(y-y_o)^2}.$

$$u(x,y)=\frac{(x-x_o)}{(x-x_o)^2+(y-y_o)^2}\quad\rightarrow\quad\begin{cases}\dfrac{\partial u}{\partial x}=-\dfrac{(x-x_o)^2-(y-y_o)^2}{[(x-x_o)^2+(y-y_o)^2]^2},\\[2ex]\dfrac{\partial u}{\partial y}=\dfrac{-2(x-x_o)(y-y_o)}{[(x-x_o)^2+(y-y_o)^2]^2},\end{cases}$$

$$v(x,y)=-\frac{(y-y_o)}{(x-x_o)^2+(y-y_o)^2}\quad\rightarrow\quad\begin{cases}\dfrac{\partial v}{\partial x}=\dfrac{2(x-x_o)(y-y_o)}{[(x-x_o)^2+(y-y_o)^2]^2},\\[2ex]\dfrac{\partial v}{\partial y}=-\dfrac{(x-x_o)^2-(y-y_o)^2}{[(x-x_o)^2+(y-y_o)^2]^2}.\end{cases}$$

Clearly, $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$. The Cauchy-Riemann conditions are thus satisfied. The only singularity of this function is at $z=z_o$, where $f_1(z)$ is not defined; everywhere else the function is well-defined and has a derivative. We conclude that $f_1(z)$ is analytic everywhere in the complex plane except at the single point $z=z_o$.

b) $f_2(z)=\exp(z^2)=\exp[(x^2-y^2)+2\mathrm{i}xy]=\exp(x^2-y^2)\cos(2xy)+\mathrm{i}\exp(x^2-y^2)\sin(2xy).$

$$u(x,y)=\exp(x^2-y^2)\cos(2xy)\rightarrow\begin{cases}\partial u/\partial x=2x\exp(x^2-y^2)\cos(2xy)-2y\exp(x^2-y^2)\sin(2xy),\\ \partial u/\partial y=-2y\exp(x^2-y^2)\cos(2xy)-2x\exp(x^2-y^2)\sin(2xy),\end{cases}$$

$$v(x,y)=\exp(x^2-y^2)\sin(2xy)\rightarrow\begin{cases}\partial v/\partial x=2x\exp(x^2-y^2)\sin(2xy)+2y\exp(x^2-y^2)\cos(2xy),\\ \partial v/\partial y=-2y\exp(x^2-y^2)\sin(2xy)+2x\exp(x^2-y^2)\cos(2xy).\end{cases}$$

Clearly, $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$. The Cauchy-Riemann conditions are thus satisfied. The function $f_2(z)$ has no singularities; it is defined everywhere, and has a derivative at each and every point z. We conclude that $f_2(z)$ is analytic everywhere in the complex plane.