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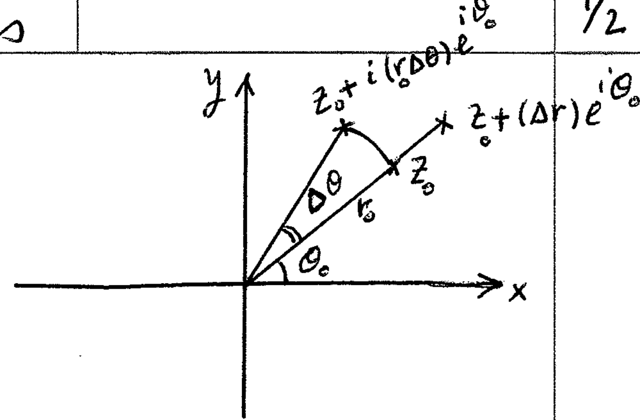
Problem 5)



a) Moving in the radial direction by Δr, the derivative of $f(z)$ at z_0 is given by:

$$f'(z_0)=\frac{f(z_0+\Delta r\, e^{i\theta_0})-f(z_0)}{\Delta r\, e^{i\theta_0}}=\frac{u(r_0+\Delta r,\theta_0)+i v(r_0+\Delta r,\theta_0)-u(r_0,\theta_0)-i v(r_0,\theta_0)}{\Delta r\, e^{i\theta_0}}$$

$$=e^{-i\theta_0}\left(\frac{\partial u(r,\theta)}{\partial r}+i\frac{\partial v(r,\theta)}{\partial r}\right)\Bigg|_{(r_0,\theta_0)}.$$

Next, we move in the azimuthal direction by $\Delta\theta$. The derivative of $f(z)$ at z_0 is given by:

$$f'(z_0)=\frac{f(z_0+i r_0\Delta\theta\, e^{i\theta_0})-f(z_0)}{i r_0\Delta\theta\, e^{i\theta_0}}=\frac{u(r_0,\theta_0+\Delta\theta)+i v(r_0,\theta_0+\Delta\theta)-u(r_0,\theta_0)-i v(r_0,\theta_0)}{i r_0\Delta\theta\, e^{i\theta_0}}$$

$$=\frac{1}{i r_0 e^{i\theta_0}}\left(\frac{\partial u(r,\theta)}{\partial\theta}+i\frac{\partial v(r,\theta)}{\partial\theta}\right)\Bigg|_{(r_0,\theta_0)}=e^{-i\theta_0}\left(\frac{1}{r_0}\frac{\partial v}{\partial\theta}-i\frac{1}{r_0}\frac{\partial u}{\partial\theta}\right)\Bigg|_{(r_0,\theta_0)}$$

The derivatives obtained by these two methods must be the same, if the function is to be differentiable at $z=z_0$. Therefore,

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial\theta}\quad\text{and}\quad\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial\theta}\quad\leftarrow\text{Cauchy-Riemann Conditions in Polar Coordinates.}$$

b) $f(z)=z^{1/2}=(re^{i\theta})^{1/2}=\sqrt{r}\,e^{i\theta/2};\qquad r\geq 0,\ 0\leq\theta<2\pi.$

$$\begin{cases}u(r,\theta)=\sqrt{r}\cos\theta/2 \Rightarrow \dfrac{\partial u}{\partial r}=\dfrac{1}{2\sqrt{r}}\cos\theta/2,\ \ \dfrac{\partial u}{\partial\theta}=-\dfrac{1}{2}\sqrt{r}\sin\theta/2\\[2ex] v(r,\theta)=\sqrt{r}\sin\theta/2 \Rightarrow \dfrac{\partial v}{\partial r}=\dfrac{1}{2\sqrt{r}}\sin\theta/2;\ \ \dfrac{\partial v}{\partial\theta}=\dfrac{1}{2}\sqrt{r}\cos\theta/2.\end{cases}$$

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The above derivatives are valid everywhere except at the origin, where $r=0$, and on the positive real axis, where $\theta=0$, the reason being that $u(r,\theta)$ and $v(r,\theta)$ are discontinuous on the positive real-axis and, therefore, can't have a derivative there.

Checking the Cauchy-Riemann Conditions:

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \Rightarrow \dfrac{1}{2\sqrt{r}} \cos \theta/2 = \dfrac{\sqrt{r}}{2r} \cos \theta/2 . \quad \checkmark \\[2ex] \dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \Rightarrow \dfrac{1}{2\sqrt{r}} \sin \theta/2 = +\dfrac{\sqrt{r}}{2r} \sin \theta/2 . \quad \checkmark \end{cases}$$

c) The branch-cut may be taken along any line that starts at the origin and goes to infinity. In the above discussion this line was taken to be the positive real axis.