Problem 5)

a) $D\frac{\partial^2 T(x,t)}{\partial x^2}=\frac{\partial T(x,t)}{\partial t}$. Here $D=K/C$ is the diffusivity of the rod's material (units: cm^2/sec), K is thermal conductivity (units: Joule/cm/sec/°C), and C is the specific heat (units: Joule/cm^3/°C).

b) $T(x,t)=f(x)g(t) \quad \rightarrow \quad Df''(x)g(t)=f(x)g'(t) \quad \rightarrow \quad D\frac{f''(x)}{f(x)}=\frac{g'(t)}{g(t)}=-\alpha.$

The separation constant must be negative, otherwise $g(t)$ will become an exponentially growing function of time, whereas on physical grounds we expect that, in the absence of an external heat source, the temperatures must decline in time. Solving the above equations, we find

$$g'(t)=-\alpha g(t) \quad \rightarrow \quad g(t)=A_0\exp(-\alpha t).$$

$$f''(x)+(\alpha/D)f(x)=0 \quad \rightarrow \quad f(x)=A_1\exp(\mathrm{i}\sqrt{\alpha/D}\,x)+A_2\exp(-\mathrm{i}\sqrt{\alpha/D}\,x).$$

The general form of separable solutions, therefore, is

$$T(x,t;\alpha)=\left[A_1(\alpha)\exp\left(\mathrm{i}\sqrt{\alpha/D}\,x\right)+A_2(\alpha)\exp\left(-\mathrm{i}\sqrt{\alpha/D}\,x\right)\right]\exp(-\alpha t).$$

Note in the above equation that we have emphasized the dependence of the coefficients A_1 and A_2 on the separation variable α. To get the above solution to resemble a Fourier kernel, we replace the parameter α with a new parameter defined as $2\pi s=\sqrt{\alpha/D}$. Note that, like α, the parameter s is real-valued and positive at this point. Both α and s may assume values in the range $[0,\infty)$ in the continuum of real numbers. The above solution may, therefore, be written as follows:

$$T(x,t;s)=[A_1(s)\exp(\mathrm{i}2\pi sx)+A_2(s)\exp(-\mathrm{i}2\pi sx)]\exp(-4\pi^2 s^2 Dt).$$

c) The superposition of the above (separable) solutions is obtained by integration over s, that is,

$$\begin{aligned}T(x,t)&=\int_0^\infty T(x,t;s)ds\\&=\int_0^\infty[A_1(s)\exp(\mathrm{i}2\pi sx)+A_2(s)\exp(-\mathrm{i}2\pi sx)]\exp(-4\pi^2 s^2 Dt)ds.\end{aligned}$$

Considering that both $+s$ and $-s$ appear in the first two exponential functions in the integrand, one could eliminate one of the two complex exponentials by extending the range of integration over all possible values of s (i.e., from $-\infty$ to $+\infty$). Consequently,

$$T(x,t)=\int_{-\infty}^\infty A_1(s)\exp(\mathrm{i}2\pi sx)\exp(-4\pi^2 s^2 Dt)ds.$$

At $t=0$, the above equation yields the initial condition as $T_0(x)=\int_{-\infty}^\infty A_1(s)\exp(\mathrm{i}2\pi sx)ds$, which shows that $A_1(s)$ is the Fourier transform of $T_0(x)$. Denoting this Fourier transform by $\hat{T}_0(s)$, we will have

$$T(x,t)=\int_{-\infty}^\infty \hat{T}_0(s)\exp(\mathrm{i}2\pi sx)\exp(-4\pi^2 s^2 Dt)ds,$$

which is the same result as had been obtained previously using the Fourier transform method.