Problem 4) The function to be minimized is $S(x,y,z) = 2xz + 2yz + xy$. The constraint is $V(x,y,z) = xyz = V_o$. We must thus form the function $S + \lambda V$, where λ is the Lagrange multiplier, then set its partial derivatives with respect to x, y, and z equal to zero. We will have

$$\partial(S + \lambda V)/\partial x = 2z + y + \lambda yz = 0 \quad \rightarrow \quad y = -2z/(1 + \lambda z),$$

$$\partial(S + \lambda V)/\partial y = 2z + x + \lambda xz = 0 \quad \rightarrow \quad x = -2z/(1 + \lambda z),$$

$$\partial(S + \lambda V)/\partial z = 2x + 2y + \lambda xy = 0.$$

The first two equations yield $x = y$, which, placed into the third equation, yields $4x + \lambda x^2 = 0$. Aside from the trivial solution, $x = 0$, the only solution to this equation is $x = -4/\lambda$. We already know that $x = y$; therefore, $y = -4/\lambda$. Substituting one of these solutions into the first or second of the above equations, we find $-4/\lambda = -2z/(1 + \lambda z) \rightarrow z = -2/\lambda$.

Having found the optimum values of x, y, and z in terms of λ, we now use the constraint $V = V_o$ to determine the value of λ, as follows:

$$V(x,y,z) = xyx = -32/\lambda^3 = V_o \quad \rightarrow \quad \lambda = -2^{5/3} V_o^{-1/3}.$$

Consequently, $x_o = y_o = -4/\lambda = (2V_o)^{1/3}$ and $z_o = -2/\lambda = (V_o/4)^{1/3}$. It is now easy to verify that $V(x_o, y_o, z_o) = V_o$ and $S(x_o, y_o, z_o) = 2x_o z_o + 2y_o z_o + x_o y_o = 3(2V_o)^{2/3}$. Any other choice for x, y, and z, which would yield the same volume V_o, will inevitably result in a larger surface area.