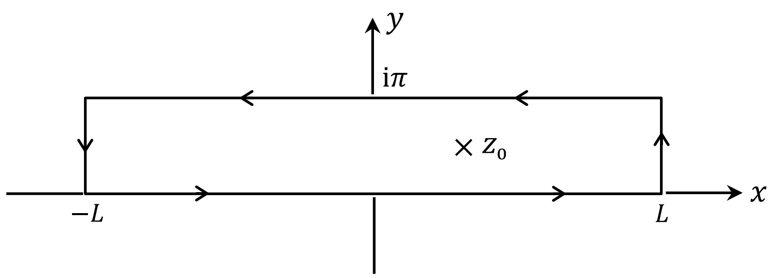
Problem 4)

$$\int_{-\infty}^{\infty}\frac{x}{a^2e^x+b^2e^{-x}}\,\mathrm{d}x=\frac{\pi\ln(b/a)}{2ab},\qquad ab>0.$$ (Gradshteyn &Ryzhik **3.417**-1)

The poles of the integrand are at $e^{2z_n}=-(b/a)^2=e^{2\ln(b/a)+\mathrm{i}(2n+1)\pi}$. Therefore, $z_n=\ln(b/a)+\mathrm{i}(n+½)\pi$. The integration contour is a rectangle of height $\mathrm{i}\pi$ and width $2L\to\infty$, as depicted below. The residue at $z_0=\ln(b/a)+\mathrm{i}(\pi/2)$ is readily evaluated, as follows:

$$\text{Residue at } z_0=\frac{z_0}{(a^2e^z+b^2e^{-z})'|_{z=z_0}}=\frac{z_0}{a^2e^{z_0}-b^2e^{-z_0}}=\frac{\ln(b/a)+\mathrm{i}(\pi/2)}{2\mathrm{i}ab}. \tag{1}$$



The loop integral is thus found to be

$$\int_{-\infty}^{\infty}\frac{x}{a^2e^x+b^2e^{-x}}\,\mathrm{d}x-\int_{-\infty}^{\infty}\frac{x+\mathrm{i}\pi}{a^2e^{(x+\mathrm{i}\pi)}+b^2e^{-(x+\mathrm{i}\pi)}}\,\mathrm{d}x=\mathrm{i}2\pi(\text{residue at } z=z_0)$$

$$\rightarrow\quad\int_{-\infty}^{\infty}\frac{2x}{a^2e^x+b^2e^{-x}}\,\mathrm{d}x+\int_{-\infty}^{\infty}\frac{\mathrm{i}\pi}{a^2e^x+b^2e^{-x}}\,\mathrm{d}x=\frac{\pi[\ln(b/a)+\mathrm{i}(\pi/2)]}{ab}. \tag{2}$$

The remaining integral, namely, that of $\mathrm{i}\pi/(a^2e^x+b^2e^{-x})$, is evaluated along similar lines, yielding

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{a^2e^x+b^2e^{-x}}-\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{a^2e^{(x+\mathrm{i}\pi)}+b^2e^{-(x+\mathrm{i}\pi)}}=2\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{a^2e^x+b^2e^{-x}}=\frac{\mathrm{i}2\pi}{a^2e^{z_0}-b^2e^{-z_0}}=\frac{\pi}{ab}$$

$$\rightarrow\quad\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{a^2e^x+b^2e^{-x}}=\frac{\pi}{2ab}. \tag{3}$$

Combining Eqs.(2) and (3), we finally arrive at

$$\int_{-\infty}^{\infty}\frac{x}{a^2e^x+b^2e^{-x}}\,\mathrm{d}x=\frac{\pi[\ln(b/a)+\mathrm{i}(\pi/2)]}{2ab}-\frac{\mathrm{i}\pi^2}{4ab}=\frac{\pi\ln(b/a)}{2ab}. \tag{4}$$