Problem 4) Let $G(s)=\int_{-\infty}^{\infty} g(x)\exp(-\mathrm{i}2\pi sx)\,\mathrm{d}x$ be the Fourier transform of $g(x)$. The inverse Fourier transform relation, namely, $g(x)=\int_{-\infty}^{\infty} G(s)\exp(\mathrm{i}2\pi sx)\,\mathrm{d}s$, when differentiated with respect to x, yields $g'(x)=\int_{-\infty}^{\infty}(\mathrm{i}2\pi s)G(s)\exp(\mathrm{i}2\pi sx)\,\mathrm{d}s$, which indicates that the Fourier transform of $g'(x)$ is $(\mathrm{i}2\pi s)G(s)$. Also, the Fourier transform of the right-hand side of the differential equation can be obtained by direct integration, as follows:

$$\begin{aligned}\mathcal{F}\{\mathrm{rect}(x)\cos(2\pi s_0 x)\} &= \int_{-½}^{½}\cos(2\pi s_0 x)\exp(-\mathrm{i}2\pi sx)\,\mathrm{d}x\\ &= ½\int_{-½}^{½}\exp[-\mathrm{i}2\pi(s-s_0)x]\,\mathrm{d}x + ½\int_{-½}^{½}\exp[-\mathrm{i}2\pi(s+s_0)x]\,\mathrm{d}x\\ &= \frac{\exp[-\mathrm{i}\pi(s-s_0)]-\exp[\mathrm{i}\pi(s-s_0)]}{-\mathrm{i}4\pi(s-s_0)} + \frac{\exp[-\mathrm{i}\pi(s+s_0)]-\exp[\mathrm{i}\pi(s+s_0)]}{-\mathrm{i}4\pi(s+s_0)}\\ &= \frac{\sin[\pi(s-s_0)]}{2\pi(s-s_0)} + \frac{\sin[\pi(s+s_0)]}{2\pi(s+s_0)}. \end{aligned} \tag{1}$$

The Fourier transform of the differential equation may thus be written as follows:

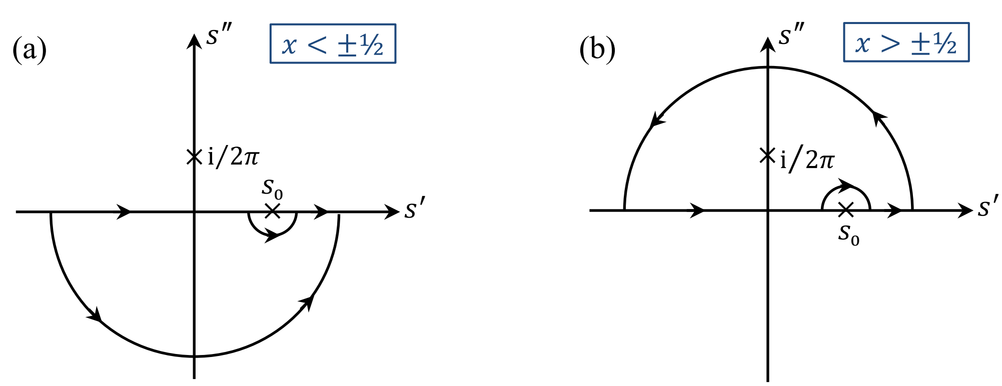
$$(\mathrm{i}2\pi s)G(s)+G(s)=\frac{\sin[\pi(s-s_0)]}{2\pi(s-s_0)}+\frac{\sin[\pi(s+s_0)]}{2\pi(s+s_0)}$$

$$\rightarrow \quad G(s)=\frac{\sin[\pi(s-s_0)]}{2\pi(s-s_0)(1+\mathrm{i}2\pi s)}+\frac{\sin[\pi(s+s_0)]}{2\pi(s+s_0)(1+\mathrm{i}2\pi s)}. \tag{2}$$

The last step is to find the inverse Fourier transform of $G(s)$. Considering that each sine function is a linear combination of two complex exponentials, we write

$$\begin{aligned} g(x)=\mathcal{F}^{-1}\{G(s)\} &= \int_{-\infty}^{\infty}\frac{\exp[\mathrm{i}\pi(s-s_0)+\mathrm{i}2\pi sx]}{\mathrm{i}4\pi(s-s_0)(1+\mathrm{i}2\pi s)}\mathrm{d}s - \int_{-\infty}^{\infty}\frac{\exp[-\mathrm{i}\pi(s-s_0)+\mathrm{i}2\pi sx]}{\mathrm{i}4\pi(s-s_0)(1+\mathrm{i}2\pi s)}\mathrm{d}s\\ &+\int_{-\infty}^{\infty}\frac{\exp[\mathrm{i}\pi(s+s_0)+\mathrm{i}2\pi sx]}{\mathrm{i}4\pi(s+s_0)(1+\mathrm{i}2\pi s)}\mathrm{d}s - \int_{-\infty}^{\infty}\frac{\exp[-\mathrm{i}\pi(s+s_0)+\mathrm{i}2\pi sx]}{\mathrm{i}4\pi(s+s_0)(1+\mathrm{i}2\pi s)}\mathrm{d}s. \end{aligned} \tag{3}$$

The four integrals in Eq.(3) should be evaluated in the complex s-plane along the contours shown in the figure below. The exponential factors appearing in the integrands are in the form of $\exp[\mathrm{i}2\pi(x\pm ½)s \pm \mathrm{i}\pi s_0]$. Depending on x being greater than or less than $\pm½$, the contours must be closed in the upper-half or lower-half of the s-plane. Each integrand has a simple pole at $s=\mathrm{i}/2\pi$, and a second (also simple) pole at either $s=s_0$ or $s=-s_0$.



The first integral in Eq.(3) should be evaluated in the lower half of the complex s-plane when $x < -½$, and in the upper half when $x > -½$, as shown in the figure. The poles of the integrand are located at $s = s_0$ and $s = \mathrm{i}/2\pi$. Therefore,

$$\int_{-\infty}^{\infty} \frac{\exp[\mathrm{i}\pi(s-s_0)+\mathrm{i}2\pi sx]}{-8\pi^2(s-s_0)[s-(\mathrm{i}/2\pi)]}\,\mathrm{d}s = \begin{cases} \dfrac{-\mathrm{i}\pi\exp(\mathrm{i}2\pi s_0 x)}{-8\pi^2[s_0-(\mathrm{i}/2\pi)]}; & x < -½, \\ \dfrac{\mathrm{i}\pi\exp(\mathrm{i}2\pi s_0 x)}{-8\pi^2[s_0-(\mathrm{i}/2\pi)]} + \dfrac{\mathrm{i}2\pi\exp\{\mathrm{i}\pi[(\mathrm{i}/2\pi)-s_0]-x\}}{-8\pi^2[(\mathrm{i}/2\pi)-s_0]}; & x > -½. \end{cases} \tag{4}$$

The second integral is evaluated in the lower half-plane when $x < ½$, and in the upper half-plane when $x > ½$. The poles of the integrand are at $s = s_0$ and $s = \mathrm{i}/2\pi$. Therefore,

$$\int_{-\infty}^{\infty} \frac{\exp[-\mathrm{i}\pi(s-s_0)+\mathrm{i}2\pi sx]}{-8\pi^2(s-s_0)[s-(\mathrm{i}/2\pi)]}\,\mathrm{d}s = \begin{cases} \dfrac{-\mathrm{i}\pi\exp(\mathrm{i}2\pi s_0 x)}{-8\pi^2[s_0-(\mathrm{i}/2\pi)]}; & x < ½, \\ \dfrac{\mathrm{i}\pi\exp(\mathrm{i}2\pi s_0 x)}{-8\pi^2[s_0-(\mathrm{i}/2\pi)]} + \dfrac{\mathrm{i}2\pi\exp\{-\mathrm{i}\pi[(\mathrm{i}/2\pi)-s_0]-x\}}{-8\pi^2[(\mathrm{i}/2\pi)-s_0]}; & x > ½. \end{cases} \tag{5}$$

The third integral is evaluated in the lower half-plane when $x < -½$, and in the upper half-plane when $x > -½$. The poles of the integrand are at $s = -s_0$ and $s = \mathrm{i}/2\pi$. Therefore,

$$\int_{-\infty}^{\infty} \frac{\exp[\mathrm{i}\pi(s+s_0)+\mathrm{i}2\pi sx]}{-8\pi^2(s+s_0)[s-(\mathrm{i}/2\pi)]}\,\mathrm{d}s = \begin{cases} \dfrac{-\mathrm{i}\pi\exp(-\mathrm{i}2\pi s_0 x)}{8\pi^2[s_0+(\mathrm{i}/2\pi)]}; & x < -½, \\ \dfrac{\mathrm{i}\pi\exp(-\mathrm{i}2\pi s_0 x)}{8\pi^2[s_0+(\mathrm{i}/2\pi)]} + \dfrac{\mathrm{i}2\pi\exp\{\mathrm{i}\pi[(\mathrm{i}/2\pi)+s_0]-x\}}{-8\pi^2[(\mathrm{i}/2\pi)+s_0]}; & x > -½. \end{cases} \tag{6}$$

The fourth integral is evaluated in the lower half-plane when $x < ½$, and in the upper half-plane when $x > ½$. The poles of the integrand are at $s = -s_0$ and $s = \mathrm{i}/2\pi$. Therefore,

$$\int_{-\infty}^{\infty} \frac{\exp[-\mathrm{i}\pi(s+s_0)+\mathrm{i}2\pi sx]}{-8\pi^2(s+s_0)[s-(\mathrm{i}/2\pi)]}\,\mathrm{d}s = \begin{cases} \dfrac{-\mathrm{i}\pi\exp(-\mathrm{i}2\pi s_0 x)}{8\pi^2[s_0+(\mathrm{i}/2\pi)]}; & x < ½, \\ \dfrac{\mathrm{i}\pi\exp(-\mathrm{i}2\pi s_0 x)}{8\pi^2[s_0+(\mathrm{i}/2\pi)]} + \dfrac{\mathrm{i}2\pi\exp\{-\mathrm{i}\pi[(\mathrm{i}/2\pi)+s_0]-x\}}{-8\pi^2[(\mathrm{i}/2\pi)+s_0]}; & x > ½. \end{cases} \tag{7}$$

Returning to Eq.(3), we now combine the integrals given in Eqs.(4)-(7) to determine $g(x)$ over the entire x-axis (from $-\infty$ to ∞), as follows:

$$g(x) = \begin{cases} 0; & x < -½, \\ \dfrac{\mathrm{i}\pi\exp(-\mathrm{i}2\pi s_0 x)}{4\pi^2[s_0+(\mathrm{i}/2\pi)]} - \dfrac{\mathrm{i}\pi\exp(\mathrm{i}2\pi s_0 x)}{4\pi^2[s_0-(\mathrm{i}/2\pi)]} + \dfrac{\exp(-x-½-\mathrm{i}\pi s_0)}{-2(1+\mathrm{i}2\pi s_0)} + \dfrac{\exp(-x-½+\mathrm{i}\pi s_0)}{-2(1-\mathrm{i}2\pi s_0)}; & |x| < ½, \\ \dfrac{\exp(-x-½-\mathrm{i}\pi s_0) - \exp(-x+½+\mathrm{i}\pi s_0)}{-2(1+\mathrm{i}2\pi s_0)} + \dfrac{\exp(-x-½+\mathrm{i}\pi s_0) - \exp(-x+½-\mathrm{i}\pi s_0)}{-2(1-\mathrm{i}2\pi s_0)}; & x > ½. \end{cases} \tag{8}$$

Further simplifications yield

$$g(x) = \begin{cases} 0; & x < -½, \\ \dfrac{\mathrm{i}\pi\{[s_0-(\mathrm{i}/2\pi)]\exp(-\mathrm{i}2\pi s_0 x)-[s_0+(\mathrm{i}/2\pi)]\exp(\mathrm{i}2\pi s_0 x)\}}{1+4\pi^2 s_0^2} - \dfrac{\exp(-x-½)[(1-\mathrm{i}2\pi s_0)\exp(-\mathrm{i}\pi s_0)+(1+\mathrm{i}2\pi s_0)\exp(\mathrm{i}\pi s_0)]}{2(1+4\pi^2 s_0^2)}; & |x| < ½ \\ -\dfrac{(1-\mathrm{i}2\pi s_0)[\exp(-x-½-\mathrm{i}\pi s_0)-\exp(-x+½+\mathrm{i}\pi s_0)]+(1+\mathrm{i}2\pi s_0)[\exp(-x-½+\mathrm{i}\pi s_0)-\exp(-x+½-\mathrm{i}\pi s_0)]}{2(1+4\pi^2 s_0^2)}; & x > ½. \end{cases}$$

$$g(x) = \begin{cases} 0; & x < -\tfrac{1}{2}, \\ \dfrac{\cos(2\pi s_0 x) + 2\pi s_0 \sin(2\pi s_0 x) - \exp(-x-\tfrac{1}{2})[\cos(\pi s_0) - 2\pi s_0 \sin(\pi s_0)]}{1+4\pi^2 s_0^2}; & |x| < \tfrac{1}{2}, \\ \dfrac{-\exp(-x-\tfrac{1}{2})[(1-\mathrm{i}2\pi s_0)\exp(-\mathrm{i}\pi s_0) + (1+\mathrm{i}2\pi s_0)\exp(\mathrm{i}\pi s_0)] + \exp(-x+\tfrac{1}{2})[(1-\mathrm{i}2\pi s_0)\exp(\mathrm{i}\pi s_0) + (1+\mathrm{i}2\pi s_0)\exp(-\mathrm{i}\pi s_0)]}{2(1+4\pi^2 s_0^2)}; & x > \tfrac{1}{2}. \end{cases}$$

$$g(x) = \begin{cases} 0; & x < -\tfrac{1}{2}, \\ \dfrac{\cos(2\pi s_0 x) + 2\pi s_0 \sin(2\pi s_0 x) - [\cos(\pi s_0) - 2\pi s_0 \sin(\pi s_0)]\exp(-x-\tfrac{1}{2})}{1+4\pi^2 s_0^2}; & |x| < \tfrac{1}{2}, \\ \dfrac{[\cos(\pi s_0) + 2\pi s_0 \sin(\pi s_0)]\exp(-x+\tfrac{1}{2}) - [\cos(\pi s_0) - 2\pi s_0 \sin(\pi s_0)]\exp(-x-\tfrac{1}{2})}{1+4\pi^2 s_0^2}; & x > \tfrac{1}{2}. \end{cases} \qquad (9)$$

As expected, the above solution is continuous at both extremes of the excitation function, namely, at $x = \pm\tfrac{1}{2}$.
