Opti 503A **Final Exam Solutions** **Spring 2017**

Problem 4)

a) $f(x,t)=\int_{-\infty}^{\infty}F(s,t)\exp(\mathrm{i}2\pi sx)\,\mathrm{d}s \quad\rightarrow\quad \frac{\partial^2 f(x,t)}{\partial x^2}=\int_{-\infty}^{\infty}(\mathrm{i}2\pi s)^2F(s,t)\exp(\mathrm{i}2\pi sx)\,\mathrm{d}s.$

Thus the Fourier transform of $\partial^2 f(x,t)/\partial x^2$ is $-(2\pi s)^2F(s,t)$. Substitution into the differential equation now yields

$$-\alpha(2\pi s)^2F(s,t)=\frac{\partial}{\partial t}F(s,t). \qquad (1)$$

b) The Fourier transform of $\exp(-\pi x^2)$ is $\exp(-\pi s^2)$. The differentiation theorem of Fourier transform theory may now be invoked to determine $F(s,t=0)$, as follows:

$$F(s,t=0)=\mathcal{F}\{f(x,t=0)\}=\mathcal{F}\{\mathrm{d}\exp(-\pi x^2)/\mathrm{d}x\}=\mathrm{i}2\pi s\exp(-\pi s^2). \qquad (2)$$

c) The solution to Eq.(1) is readily found to be

$$F(s,t)=F(s,t=0)\exp(-4\pi^2\alpha s^2 t)=\mathrm{i}2\pi s\exp(-\pi s^2)\exp(-4\pi^2\alpha s^2 t). \qquad (3)$$

d) We thus have

$$f(x,t)=\mathcal{F}^{-1}\{F(s,t)\}=\mathcal{F}^{-1}\{\mathrm{i}2\pi s\exp[-\pi(1+4\pi\alpha t)s^2]\}$$

Differentiation theorem →

$$=\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{F}^{-1}\{\exp[-\pi(\sqrt{1+4\pi\alpha t}\;s)^2]\}$$

Scaling theorem →

$$=\frac{1}{\sqrt{1+4\pi\alpha t}}\frac{\mathrm{d}}{\mathrm{d}x}\exp\left[-\pi\left(\frac{x}{\sqrt{1+4\pi\alpha t}}\right)^2\right]$$

$$=-\frac{2\pi x}{(1+4\pi\alpha t)^{3/2}}\exp\left(-\frac{\pi x^2}{1+4\pi\alpha t}\right). \qquad (4)$$