Problem 4) a) Choose a small volume element having dimensions $(\Delta r, \Delta\varphi, \Delta z)$, centered at an arbitrary point (r, φ, z) in the cylindrical coordinate system. The net heat diffusing into this volume element from adjacent regions will cause the temperature to rise in proportion to the specific heat, as follows:

$$\text{Right-hand side} \quad \kappa \frac{\partial T(r+½\Delta r,\varphi,z,t)}{\partial r}(r+½\Delta r)\Delta\varphi\Delta z - \kappa \frac{\partial T(r-½\Delta r,\varphi,z,t)}{\partial r}(r-½\Delta r)\Delta\varphi\Delta z \quad \text{Left-hand side}$$

$$\text{Front facet} \quad +\kappa \frac{\partial T(r,\varphi+½\Delta\varphi,z,t)}{r\partial\varphi}\Delta r\Delta z - \kappa \frac{\partial T(r,\varphi-½\Delta\varphi,z,t)}{r\partial\varphi}\Delta r\Delta z \quad \text{Rear facet}$$

$$\text{Top \& bottom facets} \quad +\kappa \frac{\partial T(r,\varphi,z+½\Delta z,t)}{\partial z} r\Delta r\Delta\varphi - \kappa \frac{\partial T(r,\varphi,z-½\Delta z,t)}{\partial z} r\Delta r\Delta\varphi = C\frac{\partial T(r,\varphi,z,t)}{\partial t} r\Delta r\Delta\varphi\Delta z. \quad \text{Temperature rise due to accumulated heat}$$

Dividing both sides of this equation by $Cr\Delta r\Delta\varphi\Delta z$, then allowing $\Delta r \to 0$, $\Delta\varphi \to 0$ and $\Delta z \to 0$, we find

$$(\kappa/C)\left\{\frac{\partial}{r\partial r}\left[r\frac{\partial T(r,\varphi,z,t)}{\partial r}\right] + \frac{\partial^2 T(r,\varphi,z,t)}{r^2\partial\varphi^2} + \frac{\partial^2 T(r,\varphi,z,t)}{\partial z^2}\right\} = \frac{\partial T(r,\varphi,z,t)}{\partial t}.$$

The above diffusion equation may be further simplified, as follows:

$$D\left[\frac{\partial^2 T(r,\varphi,z,t)}{\partial r^2} + \frac{\partial T(r,\varphi,z,t)}{r\partial r} + \frac{\partial^2 T(r,\varphi,z,t)}{r^2\partial\varphi^2} + \frac{\partial^2 T(r,\varphi,z,t)}{\partial z^2}\right] = \frac{\partial T(r,\varphi,z,t)}{\partial t}.$$

b) Applying the method of separation of variables, we write $T(r,\varphi,z,t) = f(r)g(\varphi)h(z)p(t)$. Substitution into the diffusion equation yields

$$D[f''(r)g(\varphi)h(z)p(t) + r^{-1}f'(r)g(\varphi)h(z)p(t) + r^{-2}f(r)g''(\varphi)h(z)p(t)$$
$$+f(r)g(\varphi)h''(z)p(t)] = f(r)g(\varphi)h(z)p'(t).$$

Dividing the above equation by $f(r)g(\varphi)h(z)p(t)$, we will have

$$D\left[\frac{f''(r)}{f(r)} + \frac{f'(r)}{rf(r)} + \frac{g''(\varphi)}{r^2 g(\varphi)} + \frac{h''(z)}{h(z)}\right] = \frac{p'(t)}{p(t)}.$$

Now, functions of different variables appearing in the preceding equation must be equal to (different) constants—because there is no other way for the equation to be satisfied. Therefore,

$$p'(t) = -\alpha p(t) \quad \rightarrow \quad p(t) = A_1\exp(-\alpha t), \quad \text{Separation constant must be negative, otherwise } p(t) \text{ will grow exponentially as } t\to\infty\text{, which is non-physical.}$$

$$h''(z) = -\beta^2 h(z) \quad \rightarrow \quad h(z) = A_2\sin(\beta z) + A_3\cos(\beta z),$$

$$g''(\varphi) = -m^2 g(\varphi) \quad \rightarrow \quad g(\varphi) = A_4\sin(m\varphi + \varphi_0), \quad \text{Separation constant must be negative to make } g(\varphi) \text{ periodic; } m \text{ must be integer to make the period a multiple of } 2\pi.$$

$$\frac{f''(r)}{f(r)} + \frac{f'(r)}{rf(r)} - \frac{m^2}{r^2} - \beta^2 = -\frac{\alpha}{D} \quad \rightarrow \quad r^2 f''(r) + rf'(r) + \{[(\alpha/D) - \beta^2]r^2 - m^2\}f(r) = 0. \quad \text{Bessel's equation}$$

In these equations, m, a non-negative integer, is the azimuthal mode-number. If m were *not* an integer, the temperature would have acquired multiple values at any given point (r,φ,z,t), as adding multiples of 2π to φ would have resulted in different values of $g(\varphi)$.

The choice of a negative separation constant $(-\beta^2)$ for $h(z)$ is not necessary. Depending on the boundary conditions, this constant may be positive or negative. For a positive separation constant β^2, the corresponding solution would be $h(z) = A_2\exp(\beta z) + A_3\exp(-\beta z)$.

c) The constant β (and also the ratio A_3/A_2) is determined by satisfying the boundary conditions at $z = z_1$ and $z = z_2$. The constant α must then be chosen such that the solution to the Bessel equation would satisfy the boundary conditions at $r = R_1$ and $r = R_2$. The general solution of the Bessel equation has the form $f(r) = A_5 J_m\big(\sqrt{(\alpha/D) \pm \beta^2}\, r\big) + A_6 Y_m\big(\sqrt{(\alpha/D) \pm \beta^2}\, r\big)$. If R_1 happens to be zero, however, the Bessel function of the second kind, Y_m, should *not* be included in the above solution, as $Y_m(r)$ diverges to infinity at $r = 0$. In general, the boundary conditions at $r = R_1$ and $r = R_2$ should be satisfied by a proper choice of α and A_6/A_5.

The general solution of the diffusion equation should now be written as a superposition (with unknown coefficients) of the eigen-functions $T(r, \varphi, z, t) = f(r)g(\varphi)h(z)p(t)$ thus obtained. The initial condition, i.e., $T(r, \varphi, z, t = 0)$, may subsequently be used to determine the remaining unknown coefficients.
