Problem 4) Separation of variables: Let $T(r,\phi,t)=f(r)g(\phi)h(t)$. Substitution into the heat diffusion equation yields:

$$D\nabla^2 T = D\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}\right)=\frac{\partial T}{\partial t}$$

$$\rightarrow \quad D[f''(r)g(\phi)h(t)+r^{-1}f'(r)g(\phi)h(t)+r^{-2}f(r)g''(\phi)h(t)]=f(r)g(\phi)h'(t).$$

Dividing both sides of the above equation by $f(r)g(\phi)h(t)$, we will have

$$D\left[\frac{f''(r)}{f(r)}+r^{-1}\frac{f'(r)}{f(r)}+r^{-2}\frac{g''(\phi)}{g(\phi)}\right]=\frac{h'(t)}{h(t)}.$$

Now, the right-hand-side of this equation is independent of r and ϕ, which must therefore be set equal to a constant. We choose the real, negative constant $-c$ for $h'(t)/h(t)$, to ensure that the solution, $h(t)=\exp(-ct)$, does *not* grow with time. Similarly, we choose the real, negative constant $-\alpha^2$ for $g''(\phi)/g(\phi)$, to ensure solutions of the form $g(\phi)=\exp(\pm \mathrm{i}\alpha\phi)$, which become periodic (with the required period of $\Delta\phi=2\pi$) when α is an integer, say, $\alpha=m$. Since the initial temperature above ambient at $t=0$ is known to be in the form of $f(r)\cos\phi$, the only acceptable solution for $g(\phi)$ is $\cos\phi$ and, therefore, $m=\alpha=1$. The remaining part of the equation is now written as follows:

$$D\left[\frac{f''(r)}{f(r)}+r^{-1}\frac{f'(r)}{f(r)}-r^{-2}\right]=-c \quad\rightarrow\quad r^2 f''(r)+rf'(r)+[(c/D)r^2-1]f(r)=0.$$

This is the Bessel equation of order 1, whose two independent solutions are $J_1(\sqrt{c/D}\,r)$ and $Y_1(\sqrt{c/D}\,r)$. Since $Y_1(\cdot)$ diverges at the center of the disk, the only acceptable solution is going to be $J_1(\sqrt{c/D}\,r)$. Moreover, at the boundary $r=R$ of the disk, the temperature is fixed at the ambient temperature T_o, which forces the solution $J_1(\sqrt{c/D}\,r)$ to go to zero at this boundary. Denoting the n^{th} zero of $J_1(\rho)$ by ρ_{1n}, we will have $\sqrt{c/D}\,R=\rho_{1n}$, which yields $c=D\rho_{1n}^2/R^2$. The complete solution of the heat diffusion equation for the present problem is thus given by

$$T(r,\phi,t)=T_0+\sum_{n=1}^{\infty}A_n J_1(\rho_{1n}r/R)\cos\phi\exp(-D\rho_{1n}^2 t/R^2).$$

To find the unknown coefficients A_n, we resort to the initial condition at $t=0$, which requires that $f(r)=\sum_{n=1}^{\infty}A_n J_1(\rho_{1n}r/R)$ in the interval $0\le r\le R$. The coefficients A_n are readily obtained using the orthogonality of the Bessel functions $J_1(\rho_{1n}r/R)$ for different values of n, that is,

$$\int_0^R rf(r)J_1(\rho_{1m}r/R)\,\mathrm{d}r=\sum_{n=1}^{\infty}A_n\int_0^R rJ_1(\rho_{1n}r/R)\,J_1(\rho_{1m}r/R)\,\mathrm{d}r \;=A_m\int_0^R rJ_1^2(\rho_{1m}r/R)\,\mathrm{d}r$$

$$\rightarrow\quad A_m=\int_0^R rf(r)J_1(\rho_{1m}r/R)\,\mathrm{d}r\Big/\int_0^R rJ_1^2(\rho_{1m}r/R)\,\mathrm{d}r.$$