Problem 3) $\displays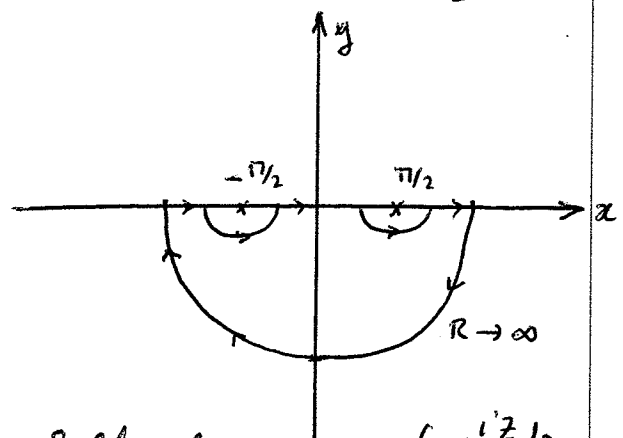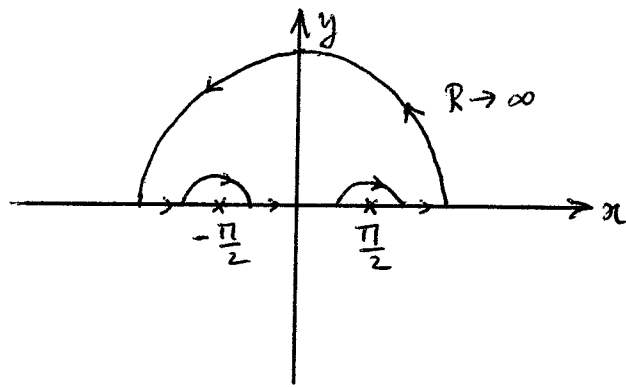tyle\int_{-\infty}^{\infty} \frac{\cos x}{\pi^2 - 4x^2}\, dx = -\frac{1}{8}\int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{(x-\frac{\pi}{2})(x+\frac{\pi}{2})} - \frac{1}{8}\int_{-\infty}^{\infty} \frac{e^{-ix}\, dx}{(x-\frac{\pi}{2})(x+\frac{\pi}{2})}$

The first integral is taken in the upper half plane, where $\oint \dfrac{e^{iz}\, dz}{z^2 - (\frac{\pi}{2})^2}$

$\to 0$ as $R \to \infty$ (Jordan's Lemma). The second integral is taken in

the lower half-plane for the same reason.

The poles at $z = \pm \pi/2$ are simple poles, but they contribute only

half of the corresponding residue to the integral, because they

are located <u>on</u> the contour (as opposed to <u>inside</u> the contour).

1st integral, residue at $z_1 = \frac{\pi}{2}$: $\left. \dfrac{e^{iz_1}}{z_1 - z_2} \right|_{z_2 = -\frac{\pi}{2}} = \dfrac{e^{i\pi/2}}{\pi} = \dfrac{i}{\pi}$. ✓

1st integral, residue at $z_2 = -\frac{\pi}{2}$: $\dfrac{e^{iz_2}}{z_2 - z_1} = \dfrac{e^{-i\pi/2}}{-\pi} = \dfrac{-i}{-\pi} = \dfrac{i}{\pi}$. ✓

2nd integral, residue at $z_1 = \frac{\pi}{2}$: $-\dfrac{e^{-iz_1}}{z_1 - z_2} = -\dfrac{e^{-i\pi/2}}{\pi} = \dfrac{i}{\pi}$. ✓

direction of travel around
small semi-circle reversed

2nd integral, residue at $z_2 = -\frac{\pi}{2}$: $-\dfrac{e^{-iz_2}}{z_2 - z_1} = -\dfrac{e^{i\pi/2}}{-\pi} = \dfrac{i}{\pi}$. ✓

Therefore, $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{\pi^2 - 4x^2}\, dx = -\frac{1}{8}(\pi i)\left(\frac{i}{\pi} + \frac{i}{\pi}\right) - \frac{1}{8}(\pi i)\left(\frac{i}{\pi} + \frac{i}{\pi}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.