

**Problem 3)**

a)  $\Gamma(1) = \int_0^{\infty} e^{-y} dy = -e^{-y} \Big|_0^{\infty} = 1.$

b)  $\Gamma(2) = \int_0^{\infty} ye^{-y} dy = -ye^{-y} \Big|_0^{\infty} + \int_0^{\infty} e^{-y} dy = 0 - 0 + 1 = 1.$  ← Integration by parts

c)  $\Gamma(x) = \int_0^{\infty} y^{x-1} e^{-y} dy = -y^{x-1} e^{-y} \Big|_0^{\infty} + \int_0^{\infty} (x-1)y^{x-2} e^{-y} dy$  ← Integration by parts  
 $= 0 - 0 + (x-1) \int_0^{\infty} y^{x-2} e^{-y} dy = (x-1)\Gamma(x-1).$

d) For an integer value of  $x$  greater than or equal to 2, namely,  $x = n \geq 2$ , we will have

$$\begin{aligned} \Gamma(n) &= (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \dots \\ &= (n-1)(n-2)(n-3) \dots 3 \times 2 \times 1 \times \Gamma(1) = (n-1)!. \end{aligned}$$

e)  $\Gamma(1/2) = \int_0^{\infty} y^{-1/2} e^{-y} dy = 2 \int_0^{\infty} e^{-x^2} dx = \sqrt{\pi}.$

Change of variable:  
 $x = y^{1/2} \rightarrow dx = \frac{1}{2}y^{-1/2} dy$

HW Problem 4

f) Using the results of parts (c) and (e) above, we may write for  $x = n + \frac{1}{2}$  where  $n \geq 1$ ,

$$\begin{aligned} \Gamma(n + \frac{1}{2}) &= (n - \frac{1}{2})\Gamma(n - \frac{1}{2}) = (n - \frac{1}{2})(n - \frac{3}{2})\Gamma(n - \frac{3}{2}) = \dots \\ &= (n - \frac{1}{2})(n - \frac{3}{2}) \dots \frac{3}{2} \times \frac{1}{2} \times \Gamma(\frac{1}{2}) = \frac{(2n-1)!!}{2^n} \sqrt{\pi}. \end{aligned}$$