Problem 3) The function to be minimized is $S(r,h) = 2\pi r^2 + 2\pi r h$. The Constraint is $V(r,h) = \pi r^2 h = V_0$. We form the function $f(r,h) = 2\pi r^2 + 2\pi r h + \lambda \pi r^2 h$, where λ is the Lagrange multiplier. We'll have:

$$\begin{cases} \dfrac{\partial f}{\partial r} = 4\pi r + 2\pi h + 2\pi \lambda r h = 0 \Rightarrow 2r + (1+\lambda r)h = 0 \Rightarrow h = -\dfrac{2r}{1+\lambda r} \\ \dfrac{\partial f}{\partial h} = 2\pi r + \lambda \pi r^2 = 0 \Rightarrow r = 0, \text{ which is unacceptable, and } r = -\dfrac{2}{\lambda}. \end{cases}$$

Substituting $r = -2/\lambda$ in the expression for h yields: $h = -\dfrac{4}{\lambda}$.

Next, we use the Constraint $\pi r^2 h = V_0$ to determine λ:

$$\pi\left(-\frac{2}{\lambda}\right)^2\left(-\frac{4}{\lambda}\right) = V_0 \Rightarrow -\frac{16\pi}{\lambda^3} = V_0 \Rightarrow \lambda^3 = -\frac{16\pi}{V_0} \Rightarrow \lambda = -2\sqrt[3]{\frac{2\pi}{V_0}}.$$

Therefore, $r = -\dfrac{2}{\lambda} = \sqrt[3]{\dfrac{V_0}{2\pi}}$ and $h = -\dfrac{4}{\lambda} = 2\sqrt[3]{\dfrac{V_0}{2\pi}}$.

It is readily verified that $\pi r^2 h = V_0$. The minimum surface area is found to be: $S(r,h) = 2\pi r^2 + 2\pi r h = 2\pi\left(\frac{V_0}{2\pi}\right)^{2/3} + 4\pi\left(\frac{V_0}{2\pi}\right)^{2/3} =$

$$6\pi\left(\frac{V_0}{2\pi}\right)^{2/3} = 3\left(2\pi V_0^2\right)^{1/3}.$$

<u>Digression</u>: For a <u>direct</u> method of calculation of this problem, set $h = \dfrac{V_0}{\pi r^2}$ into the expression for $S(r,h)$ to find: $S(r,h) = 2\pi r^2 + \dfrac{2V_0}{r}$.

The minimum of S is then found by setting its derivative w.r.t. r equal to zero, that is, $4\pi r - \dfrac{2V_0}{r^2} = 0 \Rightarrow 4\pi r^3 = 2V_0 \Rightarrow r = \sqrt[3]{\dfrac{V_0}{2\pi}}$.

Note that $\dfrac{d^2 S}{dr^2} = 4\pi + \dfrac{4V_0}{r^3}\Big|_{r=\sqrt[3]{V_0/2\pi}} = 4\pi + 8\pi = 12\pi > 0$, and, therefore, the solution obtained is in fact a minimum, not a maximum.