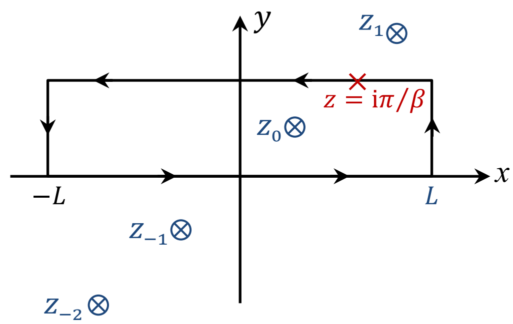
Problem 3) Taking advantage of the fact that the integrand is an even function of x, we extend the domain of integration form $(0,\infty)$ to $(-\infty,\infty)$. We also express $\cos(\alpha x)$ in terms of complex exponentials, and arrive at

$$\int_0^\infty \frac{\cos(\alpha x)}{\cosh(\beta x)}\mathrm{d}x = ¼\int_{-\infty}^\infty \frac{\exp(\mathrm{i}\alpha x)}{\cosh(\beta x)}\mathrm{d}x + ¼\int_{-\infty}^\infty \frac{\exp(-\mathrm{i}\alpha x)}{\cosh(\beta x)}\mathrm{d}x. \qquad (1)$$

To find the poles of the above integrands, we set $\cosh(\beta z)$ equal to zero, which yields $\exp(2\beta z) = -1 = \exp[\mathrm{i}(2n+1)\pi]$. Therefore, the poles are at $z_n = \mathrm{i}(n+½)\pi/\beta$, where $n = 0, \pm1, \pm2, \cdots$. The integration contour for the integrals on the right-hand side of Eq.(1) is shown on the right. Next, we evaluate $\cosh(\beta z)$ on the upper leg of the rectangular contour by writing $z = x + \mathrm{i}\eta$, where η is an arbitrary complex constant. We will have



$$\begin{aligned}\cosh(\beta z) &= ½\{\exp[\beta(x+\mathrm{i}\eta)] + \exp[-\beta(x+\mathrm{i}\eta)]\}\\ &= ½\exp(\beta x)[\cos(\beta\eta) + \mathrm{i}\sin(\beta\eta)] + ½\exp(-\beta x)[\cos(\beta\eta) - \mathrm{i}\sin(\beta\eta)]\\ &= \cosh(\beta x)\cos(\beta\eta) + \mathrm{i}\sinh(\beta x)\sin(\beta\eta). \qquad (2)\end{aligned}$$

Therefore, on the upper leg of the rectangular contour, where $z = x + \mathrm{i}(\pi/\beta)$, we have $\cosh(\beta z) = -\cosh(\beta x)$. On the vertical legs, it is not difficult to show that the integrands approach zero when $L \to \infty$. We thus have

$$\int_{-\infty}^\infty \frac{\exp(\mathrm{i}\alpha x)}{\cosh(\beta x)}\mathrm{d}x + \int_{-\infty}^\infty \frac{\exp[\mathrm{i}\alpha(x+\mathrm{i}\pi/\beta)]}{\cosh(\beta x)}\mathrm{d}x = 2\pi\mathrm{i}\left[\frac{\exp(\mathrm{i}\alpha z_0)}{\beta\sinh(\beta z_0)}\right] \leftarrow \boxed{\text{Residue at } z_0}$$

$$= 2\pi\mathrm{i}\left[\frac{\exp(-\pi\alpha/2\beta)}{\beta\sinh(\mathrm{i}\pi/2)}\right] = \left(\frac{2\pi}{\beta}\right)\exp(-\pi\alpha/2\beta). \qquad (3)$$

The above equation yields

$$\int_{-\infty}^\infty \frac{\exp(\mathrm{i}\alpha x)}{\cosh(\beta x)}\mathrm{d}x = \left(\frac{2\pi}{\beta}\right)\frac{\exp(-\pi\alpha/2\beta)}{1+\exp(-\pi\alpha/\beta)} = \frac{\pi/\beta}{\cosh(\pi\alpha/2\beta)}. \qquad (4)$$

Similarly, the second integral on the right-hand side of Eq.(1) is evaluated with the aid of the rectangular contour as follows:

$$\int_{-\infty}^\infty \frac{\exp(-\mathrm{i}\alpha x)}{\cosh(\beta x)}\mathrm{d}x + \int_{-\infty}^\infty \frac{\exp[-\mathrm{i}\alpha(x+\mathrm{i}\pi/\beta)]}{\cosh(\beta x)}\mathrm{d}x = 2\pi\mathrm{i}\left[\frac{\exp(-\mathrm{i}\alpha z_0)}{\beta\sinh(\beta z_0)}\right] \leftarrow \boxed{\text{Residue at } z_0}$$

$$= 2\pi\mathrm{i}\left[\frac{\exp(\pi\alpha/2\beta)}{\beta\sinh(\mathrm{i}\pi/2)}\right] = \left(\frac{2\pi}{\beta}\right)\exp(\pi\alpha/2\beta). \qquad (5)$$

Consequently,

$$\int_{-\infty}^\infty \frac{\exp(-\mathrm{i}\alpha x)}{\cosh(\beta x)}\mathrm{d}x = \left(\frac{2\pi}{\beta}\right)\frac{\exp(\pi\alpha/2\beta)}{1+\exp(\pi\alpha/\beta)} = \frac{\pi/\beta}{\cosh(\pi\alpha/2\beta)}. \qquad (6)$$

A final substitution into Eq.(1) yields

$$\int_0^\infty \frac{\cos(\alpha x)}{\cosh(\beta x)}\mathrm{d}x = \frac{\pi/2\beta}{\cosh(\pi\alpha/2\beta)}. \tag{7}$$