Problem 3)

a) $$\cosh(x+\mathrm{i}y) = ½\exp(x)\exp(\mathrm{i}y) + ½\exp(-x)\exp(-\mathrm{i}y)$$

$$= ½\exp(x)(\cos y + \mathrm{i}\sin y) + ½\exp(-x)(\cos y - \mathrm{i}\sin y)$$

$$= ½[\exp(x)+\exp(-x)]\cos(y) + ½\mathrm{i}[\exp(x)-\exp(-x)]\sin(y)$$

$$= \cosh(x)\cos(y) + \mathrm{i}\sinh(x)\sin(y). \qquad (1)$$

b) $\cosh(z) = 0 \quad \rightarrow \quad \exp(z)+\exp(-z) = 0 \quad \rightarrow \quad \exp(2z) = -1 = \exp[\pm\mathrm{i}(2n+1)\pi].$

Consequently, the zeros of $\cosh(z)$ are at $z_n = \pm\mathrm{i}(n+½)\pi$, where $n = 0, 1, 2, 3, \cdots$.

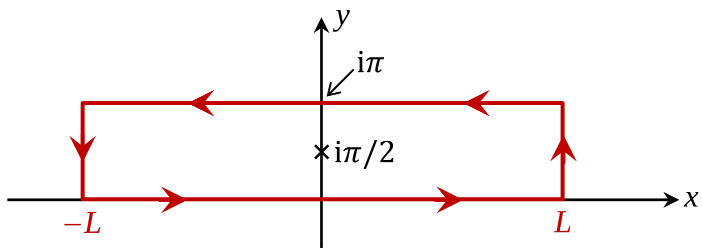
Alternatively, one could use Eq.(1) and set $\cosh(x)\cos(y) = 0$, which yields $y_n = \pm(n+½)\pi$, and $\sinh(x)\sin(y) = 0$, which yields $x = 0$. Therefore, $z_n = \pm\mathrm{i}(n+½)\pi$.

c) On the horizontal line $z = x + \mathrm{i}\pi$ within the complex z-plane, we have

$$\cosh(z) = \cosh(x)\cos(\pi) + \mathrm{i}\sinh(x)\sin(\pi) = -\cosh(x). \qquad (2)$$

d) The integration contour in the complex z-plane is shown below. Considering that, on the short vertical legs of the contour at $x = \pm L$, the integrand $1/\cosh(z)$ goes to zero when $L \rightarrow \infty$, we will have

$$\int_{-\infty}^{\infty}\frac{dx}{\cosh x} - \int_{-\infty}^{\infty}\frac{dx}{-\cosh x} = 2\pi\mathrm{i} \times [\text{Residue of } 1/\cosh(z) \text{ at } z_0 = \mathrm{i}\pi/2]. \qquad (3)$$



A Taylor series expansion of $\cosh(z)$ around the pole of the integrand at $z_0 = \mathrm{i}\pi/2$ yields

$$\cosh(z) = \cosh(z_0) + \sinh(z_0)(z-z_0) + \frac{\cosh(z_0)}{2!}(z-z_0)^2 + \cdots$$

$$= 0 + ½[\exp(\mathrm{i}\pi/2) - \exp(-\mathrm{i}\pi/2)](z-z_0) + 0 + \cdots$$

$$= \mathrm{i}(z-z_0) + \text{higher order terms.}$$

The residue of $1/\cosh(z)$ at $z_0 = \mathrm{i}\pi/2$ is thus seen to be $1/\mathrm{i}$. Therefore, Eq.(3) may be written

$$2\int_{-\infty}^{\infty}\frac{dx}{\cosh x} = 2\pi\mathrm{i}/\mathrm{i} = 2\pi \qquad \rightarrow \qquad \int_{-\infty}^{\infty}\frac{dx}{\cosh x} = \pi. \qquad (4)$$

The second integral, $\int_{-\infty}^{\infty}[x/\cosh(x)]dx$, obviously vanishes because its integrand is an odd function of x. Nevertheless, we proceed to use the Cauchy theorem on the same contour as before to obtain

$$\int_{-\infty}^{\infty} \frac{x}{\cosh x} dx - \int_{-\infty}^{\infty} \frac{x + \mathrm{i}\pi}{-\cosh x} dx = 2\pi \mathrm{i} \times [\text{Residue of } z/\cosh(z) \text{ at } z_0 = \mathrm{i}\pi/2]$$

$$\rightarrow \quad 2\int_{-\infty}^{\infty} \frac{x}{\cosh x} dx + \mathrm{i}\pi \int_{-\infty}^{\infty} \frac{1}{\cosh x} dx = 2\pi \mathrm{i} \times \pi/2 \quad \rightarrow \quad \int_{-\infty}^{\infty} \frac{x}{\cosh x} dx = 0. \tag{5}$$

As for the third integral, $\int_{-\infty}^{\infty} [x^2/\cosh(x)] dx$, we use a similar procedure, namely,

$$\int_{-\infty}^{\infty} \frac{x^2}{\cosh x} dx - \int_{-\infty}^{\infty} \frac{(x + \mathrm{i}\pi)^2}{-\cosh x} dx = 2\pi \mathrm{i} \times [\text{Residue of } z^2/\cosh(z) \text{ at } z_0 = \mathrm{i}\pi/2]$$

$$\rightarrow \; 2\int_{-\infty}^{\infty} \frac{x^2 dx}{\cosh x} + \mathrm{i}2\pi \underbrace{\int_{-\infty}^{\infty} \frac{x dx}{\cosh x}}_{0} - \pi^2 \int_{-\infty}^{\infty} \frac{dx}{\cosh x} = 2\pi \times (\mathrm{i}\pi/2)^2 \; \rightarrow \; \int_{-\infty}^{\infty} \frac{x^2 dx}{\cosh x} = \pi^3/4. \tag{6}$$