Problem 3)

a) We guess that the solution is of the form $f(x) = x^s$. substitution into the differential equation then yields

$$ax^2 s(s-1)x^{s-2} + bxsx^{s-1} + cx^s = 0 \quad \rightarrow \quad as^2 - (a-b)s + c = 0$$

$$\rightarrow \quad s_{1,2} = \frac{(a-b) \pm \sqrt{(a-b)^2 - 4ac}}{2a} \quad \rightarrow \quad f(x) = A_1 x^{s_1} + A_2 x^{s_2}.$$

b) When s_1 happens to be equal to s_2, the above method yields only one solution. This happens when the expression under the square-root vanishes, that is, $(a-b)^2 = 4ac$, or, $b = a \pm 2\sqrt{ac}$.

c)

$$f(x) = A_0 \frac{x^{s_1} - x^{s_2}}{s_1 - s_2} = A_0 x^{s_2} \left(\frac{x^{s_1 - s_2} - 1}{s_1 - s_2} \right) = A_0 x^{s_2} \left[\frac{e^{(s_1 - s_2)\ln x} - 1}{s_1 - s_2} \right]$$

$$= A_0 x^{s_2} \left[\frac{1 + (s_1 - s_2)\ln x + ½(s_1 - s_2)^2 \ln^2 x + \cdots - 1}{s_1 - s_2} \right]$$

$$= A_0 x^{s_2} [\ln x + ½(s_1 - s_2) \ln^2 x + \cdots]$$

Thus, in the limit when $s_1 \rightarrow s_2 = s$, we will have $f(x) \rightarrow A_0 x^s \ln x$. The general solution of the equi-dimensional differential equation in the special case when $b = a \pm 2\sqrt{ac}$ may thus be written as $f(x) = (A_0 + A_1 \ln x)x^s$.

d) $f(x) = x^s \ln x \;\; \rightarrow f'(x) = (1 + s\ln x)x^{s-1} \;\; \rightarrow \;\; f''(x) = [(2s-1) + s(s-1)\ln x]x^{s-2}$.

Therefore,

$$ax^2 f''(x) + bxf'(x) + cf(x) = a[(2s-1) + s(s-1)\ln x]x^s + b(1 + s\ln x)x^s + c(\ln x)x^s$$

$$= \left\{ 2a\left(s - \frac{a-b}{2a}\right) + [as^2 - (a-b)s + c]\ln x \right\} x^s.$$

In the special case when $b = a \pm 2\sqrt{ac}$, we have $s = s_1 = s_2 = (a-b)/(2a) = \mp\sqrt{c/a}$. Also, the coefficient of $\ln x$ in the above expression may be written as $a(s \pm \sqrt{c/a})^2$. Both terms of the expression thus vanish, confirming $f(x) = x^s \ln x$ as a solution of the differential equation.