Problem 3)

$$f(x)=x^s\sum_{k=0}^{\infty}A_kx^k \quad\rightarrow\quad f'(x)=\sum_{k=0}^{\infty}(k+s)A_kx^{k+s-1} \quad\rightarrow\quad f''(x)=\sum_{k=0}^{\infty}(k+s)(k+s-1)A_kx^{k+s-2}.$$

The differential equation may now be written

$$\begin{aligned}
&x^2f''(x)+xf'(x)-(x^2+p^2)f(x)\\
&=\sum_{k=0}^{\infty}(k+s)(k+s-1)A_kx^{k+s}+\sum_{k=0}^{\infty}(k+s)A_kx^{k+s}-(x^2+p^2)\sum_{k=0}^{\infty}A_kx^{k+s}\\
&=\sum_{k=0}^{\infty}[(k+s)^2-p^2]A_kx^{k+s}-\sum_{k=0}^{\infty}A_kx^{k+s+2}\\
&=(s^2-p^2)A_0x^s+[(s+1)^2-p^2]A_1x^{s+1}+\sum_{k=2}^{\infty}[(k+s)^2-p^2]A_kx^{k+s}-\sum_{k=2}^{\infty}A_{k-2}x^{k+s}\\
&=(s^2-p^2)A_0x^s+[(s+1)^2-p^2]A_1x^{s+1}+\sum_{k=2}^{\infty}[(k+s-p)(k+s+p)A_k-A_{k-2}]x^{k+s}=0.
\end{aligned}$$

Indicial equations:

i) $s^2-p^2=0 \quad\rightarrow\quad s_1=p,\quad s_2=-p,\quad A_0=\text{arbitrary},\quad A_1=0.$

ii) $(s+1)^2-p^2=0 \quad\rightarrow\quad s_3=p-1,\quad s_4=-p-1,\quad A_0=0,\quad A_1=\text{arbitrary}.$

Recursion relation: $(k+s-p)(k+s+p)A_k-A_{k-2}=0 \quad\rightarrow\quad A_k=\dfrac{A_{k-2}}{(k+s-p)(k+s+p)}.$

Next we must determine the coefficients A_k for each of the four values of s.

a) $s_1=p:\quad A_1=A_3=A_5=\ldots=0;\quad A_2=\dfrac{A_0}{2(2+2p)}=\dfrac{A_0}{2^2(p+1)};$

$$A_4=\frac{A_2}{4(4+2p)}=\frac{A_0}{2^4\cdot1\cdot2\cdot(p+1)(p+2)};$$

$$A_6=\frac{A_4}{6(6+2p)}=\frac{A_0}{2^6\cdot1\cdot2\cdot3\cdot(p+1)(p+2)(p+3)};$$

$$\vdots$$

$$A_{2k}=\frac{p!A_0}{2^{2k}k!(p+k)!}.$$

Therefore, $f(x)=x^p\sum_{k=0}^{\infty}\dfrac{p!A_0x^{2k}}{2^{2k}k!(p+k)!} \quad\rightarrow\quad f(x)=2^p p!A_0\sum_{k=0}^{\infty}\dfrac{(x/2)^{2k+p}}{k!(p+k)!}.$

The coefficient $2^p p!A_0$, being a constant (i.e., independent of x and k), may be dropped. The remaining $f(x)$ is generally written as $I_p(x)$ and referred to as the *modified Bessel function of the first kind, order p*.

b) $s_2 = -p$: $A_1 = A_3 = A_5 = \ldots = 0$; $A_2 = \dfrac{A_0}{2(2-2p)} = \dfrac{A_0}{2^2(1-p)}$;

$$A_4 = \frac{A_2}{4(4-2p)} = \frac{A_0}{2^4 \cdot 1 \cdot 2 \cdot (1-p)(2-p)};$$

$$A_6 = \frac{A_4}{6(6-2p)} = \frac{A_0}{2^6 \cdot 1 \cdot 2 \cdot 3 \cdot (1-p)(2-p)(3-p)};$$

$$\vdots$$

$$A_{2k} = \frac{(-p)!\,A_0}{2^{2k}k!(k-p)!}.$$

When x is not a positive integer, $x!$ is defined in terms of the Gamma function, namely, $x! = \Gamma(x+1)$.

Therefore, $f(x) = x^{-p}\sum_{k=0}^{\infty}\dfrac{(-p)!\,A_0\,x^{2k}}{2^{2k}k!\,(k-p)!} \quad \rightarrow \quad f(x) = 2^{-p}(-p)!\,A_0\sum_{k=0}^{\infty}\dfrac{(x/2)^{2k-p}}{k!\,(k-p)!}.$

The coefficient $2^{-p}(-p)!A_0$, being a constant (i.e., independent of x and k), may be dropped. The remaining $f(x)$ is written as $I_{-p}(x)$ and referred to as the *modified Bessel function of the first kind, order* $-p$. When p is non-integer, $I_p(x)$ and $I_{-p}(x)$ are the two independent solutions of the modified Bessel equation.

If p happens to be an integer, the solution obtained above for $s_2=-p$ must be re-examined, as A_{2k} goes to infinity for $k \geq p$. A careful examination of the recursion relation $A_{k-2} = k(k-2p)A_k$ reveals that the first few coefficients A_0, A_2, $A_4, \ldots, A_{2(p-1)}$ must be zero in this case. (Start with $k=2p$ and compute the coefficients A_{k-2}, $A_{k-4}, \ldots, A_0$ in declining order.) The remaining terms, however, follow the same pattern as the previous case of $s_1=p$, in which case, the solution associated with s_2 becomes the same as that obtained for s_1, namely, $I_p(x)$. Thus, for integer p, the method of Frobenius yields only one of the two independent solutions of the modified Bessel equation. A more complex technique must be used to derive the second solution.

c, d) The solutions for $s_3=p-1$ and $s_4=-p-1$ turn out to be the same as those obtained for s_1 and s_2, respectively. We will not go into the details, as the procedure is essentially the same as before.