Problem 3) Let

$$f(x)=x^s\sum_{n=0}^{\infty}a_nx^n \quad\to\quad f'(x)=\sum_{n=0}^{\infty}(n+s)a_nx^{n+s-1} \quad\to\quad f''(x)=\sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s-2}.$$

Substitution of the above expressions into the differential equation $\frac{\mathrm{d}^2f(x)}{\mathrm{d}x^2}+2\frac{\mathrm{d}f(x)}{\mathrm{d}x}+f(x)=0$ yields

$$\sum_{n=0}^{\infty}(n+s)(n+s-1)a_nx^{n+s-2}+2\sum_{n=0}^{\infty}(n+s)a_nx^{n+s-1}+\sum_{n=0}^{\infty}a_nx^{n+s}=0$$

$$\to\quad s(s-1)a_0x^{s-2}+s(s+1)a_1x^{s-1}+\sum_{n=2}^{\infty}(n+s)(n+s-1)a_nx^{n+s-2}$$

$$+2sa_0x^{s-1}+2\sum_{n=1}^{\infty}(n+s)a_nx^{n+s-1}+\sum_{n=0}^{\infty}a_nx^{n+s}=0$$

$$\to\quad s(s-1)a_0x^{s-2}+[s(s+1)a_1+2sa_0]x^{s-1}$$

$$+\sum_{k=0}^{\infty}(k+s+2)(k+s+1)a_{k+2}x^{k+s}+2\sum_{k=0}^{\infty}(k+s+1)a_{k+1}x^{k+s}+\sum_{k=0}^{\infty}a_kx^{k+s}=0.$$

Thus the indicial equations are $s(s-1)a_0=0$ and $s[(s+1)a_1+2a_0]=0$, whose solutions are readily found to be

Case 1) $s_1=0$, a_0 and a_1 arbitrary;
Case 2) $s_2=1$, a_0 arbitrary, $a_1=-a_0$;
Case 3) $s_3=-1$, a_1 arbitrary, $a_0=0$.

The recursion relation is given by

$$(k+s+2)(k+s+1)a_{k+2}+2(k+s+1)a_{k+1}+a_k=0 \quad\to\quad a_{k+2}=-\frac{2a_{k+1}}{k+s+2}-\frac{a_k}{(k+s+1)(k+s+2)}.$$

Case 1) $s_1=0$, a_0 and a_1 arbitrary, and $a_{k+2}=-\frac{2a_{k+1}}{k+2}-\frac{a_k}{(k+1)(k+2)}$. Therefore,

$$k=0:\quad a_2=-\frac{a_0}{2}-a_1.$$

$$k=1:\quad a_3=-\frac{2a_2}{3}-\frac{a_1}{6}=\frac{a_0}{3}+\frac{a_1}{2!}.$$

$$k=2:\quad a_4=-\frac{2a_3}{4}-\frac{a_2}{12}=-\frac{a_0}{8}-\frac{a_1}{3!}.$$

$$k=3: \quad a_5 = -\frac{2a_4}{5} - \frac{a_3}{20} = \frac{a_0}{30} + \frac{a_1}{4!}.$$

It thus appears that $a_n = (-1)^{n-1}\left[\frac{(n-1)a_0}{n!} + \frac{a_1}{(n-1)!}\right]$, a conjecture confirmed by substitution into the recursion relation. The solution of the differential equation in this case will be

$$f(x) = x^{s_1}\left\{a_0 + a_1 x + \sum_{n=2}^{\infty}(-1)^{n-1}\left[\frac{(n-1)a_0}{n!} + \frac{a_1}{(n-1)!}\right]x^n\right\} = a_0\left[1 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(n-1)\,x^n}{n!}\right] + a_1 x\exp(-x).$$

The differential equation is seen to have two independent solutions, one in the form of $x\exp(-x)$, the other in the form of the function that multiplies a_0. The latter solution may be further simplified as follows:

$$1 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(n-1)\,x^n}{n!} = 1 + \sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n!} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{(n-1)!} = \exp(-x) + x\exp(-x).$$

Thus the independent solutions of the equation may as well be $\exp(-x)$ and $x\exp(-x)$.

Case 2) $s_2 = 1$, a_0 arbitrary, $a_1 = -a_0$, and $a_{k+2} = -\frac{2a_{k+1}}{k+3} - \frac{a_k}{(k+2)(k+3)}$. Therefore,

$$k=0: \quad a_2 = -\frac{2a_1}{3} - \frac{a_0}{6} = \frac{2a_0}{3} - \frac{a_0}{6} = \frac{a_0}{2!}.$$

$$k=1: \quad a_3 = -\frac{2a_2}{4} - \frac{a_1}{12} = -\frac{a_0}{4} + \frac{a_0}{12} = -\frac{a_0}{3!}.$$

$$k=2: \quad a_4 = -\frac{2a_3}{5} - \frac{a_2}{20} = \frac{a_0}{15} - \frac{a_0}{40} = \frac{a_0}{4!}.$$

It thus appears that $a_n = (-1)^n a_0/n!$, a conjecture confirmed by substitution into the recursion relation. The solution of the differential equation in this case will be

$$f(x) = x^{s_2}\sum_{n=0}^{\infty}\frac{(-1)^n a_0}{n!}x^n = a_0 x\exp(-x).$$

Case 3) $s_3 = -1$, $a_0 = 0$, a_1 arbitrary, and $k(k+1)a_{k+2} = -2ka_{k+1} - a_k$.

$$k=0: \quad a_2 = \text{arbitrary}.$$

$$k=1: \quad a_3 = -\frac{2a_2}{2} - \frac{a_1}{2} = -\frac{a_1}{2} - a_2.$$

$$k=2: \quad a_4 = -\frac{2a_3}{3} - \frac{a_2}{6} = \frac{a_1}{3} + \frac{a_2}{2!}.$$

$$k=3: \quad a_5 = -\frac{2a_4}{4} - \frac{a_3}{12} = -\frac{a_1}{8} - \frac{a_2}{3!}.$$

$$k=4: \quad a_6 = -\frac{2a_5}{5} - \frac{a_4}{20} = \frac{a_1}{30} + \frac{a_2}{4!}.$$

It thus appears that $a_{n+2} = (-1)^n \left[\frac{n a_1}{(n+1)!} + \frac{a_2}{n!} \right]$, a conjecture confirmed by substitution into the recursion relation. The solution of the differential equation in this case will be

$$f(x) = x^{s_3} \left\{ a_1 x + a_2 x^2 + \sum_{n=1}^{\infty} (-1)^n \left[\frac{n a_1}{(n+1)!} + \frac{a_2}{n!} \right] x^{n+2} \right\} = a_1 \left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{n x^{n+1}}{(n+1)!} \right] + a_2 x \exp(-x).$$

Note that the second term is a solution already obtained in previous cases. If we set $a_2 = -a_1$, the above $f(x)$ becomes the second independent solution, as follows:

$$f(x) = a_1 - a_1 x + \sum_{n=1}^{\infty} (-1)^n \left[\frac{n a_1}{(n+1)!} - \frac{a_1}{n!} \right] x^{n+1} = a_1 \left[1 - x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{(n+1)!} \right] = a_1 \exp(-x).$$
