Problem 2) a) On the large circle, $f(z) = z^\lambda = R^\lambda e^{\mathrm{i}\lambda\varphi}$, with $0 \le \varphi < 2\pi$. Similarly, on the small circle, $f(z) = \varepsilon^\lambda e^{\mathrm{i}\lambda\varphi}$, again with $0 \le \varphi < 2\pi$. On the straight-line-segment immediately above the real axis, $\varphi = 0$ and $f(z) = x^\lambda$, with $\varepsilon \le x \le R$. And on the straight-line-segment immediately below the real axis, $\varphi = 2\pi$ and $f(z) = x^\lambda e^{\mathrm{i}2\pi\lambda}$, again with $\varepsilon \le x \le R$.

b)
$$\oint_{\text{circle}} z^\lambda \mathrm{d}z = \int_{\varphi=0}^{2\pi} R^\lambda e^{\mathrm{i}\lambda\varphi} \underbrace{(\mathrm{i}Re^{\mathrm{i}\varphi})\mathrm{d}\varphi}_{\mathrm{d}z} = \mathrm{i}R^{\lambda+1}\int_{\varphi=0}^{2\pi} e^{\mathrm{i}(\lambda+1)\varphi}\mathrm{d}\varphi$$
$$= \begin{cases} 2\pi\mathrm{i}, & \lambda = -1; \\ R^{\lambda+1}[e^{\mathrm{i}2\pi(\lambda+1)} - 1]/(\lambda+1), & \lambda \ne -1. \end{cases} \tag{1}$$

The integral around the circle of radius R is thus seen to be nonzero, unless λ is an integer (positive, zero, or negative) other than -1. Note in the above equation that $e^{\mathrm{i}2\pi(\lambda+1)}$ can be replaced by $e^{\mathrm{i}2\pi\lambda}$, simply because $e^{\mathrm{i}2\pi} = 1$.

c) The equation for the large circle obtained in part (b) can also be used for the small circle of radius ε, provided that the sign of the integral is reversed (because the direction of travel along the small circle is opposite to that around the large circle). We thus have

$$\oint_{\text{small circle}} z^\lambda \mathrm{d}z = \begin{cases} -2\pi\mathrm{i}, & \lambda = -1; \\ \varepsilon^{\lambda+1}(1 - e^{\mathrm{i}2\pi\lambda})/(\lambda+1), & \lambda \ne -1. \end{cases} \tag{2}$$

If λ happens to be an integer (positive, zero, or negative) other than -1, no matter how small the value of ε may be, the integral around the small circle will be zero (because $e^{\mathrm{i}2\pi\lambda} = 1$). Also, for non-integer as well as complex values of λ having $\lambda' > -1$, the integral around the small circle tends toward zero as $\varepsilon \to 0$; the reason is that $\varepsilon^{\lambda+1} = \varepsilon^{\lambda'+1}\varepsilon^{\mathrm{i}\lambda''}$ and, so long as $\lambda' + 1$ remains positive, $\varepsilon^{\lambda'+1}$ vanishes in the limit when $\varepsilon \to 0$. This is remarkable, considering that for $-1 < \lambda' < 0$, the origin at $z = 0$ is a singular point of $f(z)$.

On the straight-line-segment immediately above the branch-cut, noting that $z^\lambda = x^\lambda$, we evaluate the integral from $x = \varepsilon$ to $x = R$ as follows:

$$\int_\varepsilon^R x^\lambda \mathrm{d}x = \begin{cases} \ln R - \ln\varepsilon, & \lambda = -1; \\ (R^{\lambda+1} - \varepsilon^{\lambda+1})/(\lambda+1), & \lambda \ne -1. \end{cases} \tag{3}$$

Similarly, on the straight-line-segment immediately below the branch-cut, where $z^\lambda = (xe^{\mathrm{i}2\pi})^\lambda$, we evaluate the integral from $x = \varepsilon$ to $x = R$ as follows:

$$\int_\varepsilon^R (xe^{\mathrm{i}2\pi})^\lambda \mathrm{d}x = \begin{cases} (\ln R - \ln\varepsilon)\cancelto{1}{e^{-\mathrm{i}2\pi}}, & \lambda = -1; \\ (R^{\lambda+1} - \varepsilon^{\lambda+1})e^{\mathrm{i}2\pi\lambda}/(\lambda+1), & \lambda \ne -1. \end{cases} \tag{4}$$

Note that, for $\lambda = -1$, both integrals along the straight-line-segments diverge to infinity when $\varepsilon \to 0$ (because $\ln\varepsilon \to -\infty$). However, considering that these line-segments are traversed in opposite directions, their contributions to the overall loop integral cancel out. If λ happens to be an integer (positive, zero, or negative), then $e^{\mathrm{i}2\pi\lambda}$ appearing in Eq.(4) will be equal to 1, in which case the contributions of the two line-segments in Eqs.(3) and (4) cancel out again. (When λ is a negative integer, both integrals in Eqs.(3) and (4) diverge to infinity, but they cancel out nonetheless.) For non-integer λ, given that $e^{\mathrm{i}2\pi\lambda} \ne 1$, the two line-segments do not cancel out.

d) When $\lambda = -1$, the contributions of the two straight-line-segments given in Eqs.(3) and (4) cancel out, since they are being traversed in opposite directions. The contribution of the small circle is $-2\pi\mathrm{i}$, which cancels out the contribution of the large circle; see Eqs.(1) and (2). The overall loop integral thus vanishes, as it should in accordance with the Cauchy-Goursat theorem.

In the case of $\lambda \neq -1$, adding the contribution of the small circle of radius ε to those of the straight lines immediately above and below the branch-cut yields

$$\int_{3\text{ segments}} z^{\lambda}\mathrm{d}z = \frac{(R^{\lambda+1} - \varepsilon^{\lambda+1}) + \varepsilon^{\lambda+1}(1 - e^{\mathrm{i}2\pi\lambda}) - (R^{\lambda+1} - \varepsilon^{\lambda+1})e^{\mathrm{i}2\pi\lambda}}{\lambda+1} = \frac{R^{\lambda+1}(1 - e^{\mathrm{i}2\pi\lambda})}{\lambda+1}. \quad (5)$$

Thus, this integral cancels the one given by Eq.(1), confirming once again that the overall loop integral equals zero.

Digression: One way to demonstrate that $f(z) = f(re^{\mathrm{i}\varphi}) = r^{\lambda}e^{\mathrm{i}\lambda\varphi}$ is analytic everywhere (except, of course, at the branch-point $z = 0$ and on the branch-cut) is to compute the change $\Delta f(z)$ in $f(z)$ when z changes slightly from $re^{\mathrm{i}\varphi}$ by $\Delta z = (\Delta r + \mathrm{i}r\Delta\varphi)e^{\mathrm{i}\varphi}$. We will have

$$\Delta f(z) = (\partial_r f)\Delta r + (\partial_{\varphi} f)\Delta\varphi = \lambda r^{\lambda-1}e^{\mathrm{i}\lambda\varphi}\Delta r + \mathrm{i}\lambda r^{\lambda}e^{\mathrm{i}\lambda\varphi}\Delta\varphi = \lambda r^{\lambda-1}e^{\mathrm{i}\lambda\varphi}(\Delta r + \mathrm{i}r\Delta\varphi). \quad (6)$$

The local derivative of $f(z)$ is thus seen to be

$$f'(z) = lim_{(\Delta r,\Delta\varphi)\to 0}\frac{\Delta f(z)}{\Delta z} = \frac{\lambda r^{\lambda-1}e^{\mathrm{i}\lambda\varphi}(\Delta r + \mathrm{i}r\Delta\varphi)}{(\Delta r + \mathrm{i}r\Delta\varphi)e^{\mathrm{i}\varphi}} = \lambda r^{\lambda-1}e^{\mathrm{i}(\lambda-1)\varphi} = \lambda z^{\lambda-1}. \quad (7)$$

The above derivative is clearly independent of the direction of Δz in the complex z-plane. The derivative exists everywhere except at the branch-point $(z = 0)$ and on the branch-cut, where the phase angle φ undergoes a 2π jump.